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Minority Voting and Long-term Decisions*

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Abstract

In this paper we propose minority voting as a scheme that can partially protect individuals from the risk of repeated exploitation. We consider a committee that meets twice to decide about projects where the first-period project may have a long-lasting impact. In the first period a simple open majority voting scheme takes place. Voting splits the committee into three groups: voting winners, voting losers, and absentees. Under minority voting only voting losers keep the voting right in the second period. We show that as soon as absolute risk aversion exceeds a threshold value minority voting is superior to repeated application of the simple majority rule.

Keywords: voting, minority, durable decision, risk aversion, tyranny of majority rules

JEL Classification: D7

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1 Introduction

Suppose a society or committee takes several project decisions in series. When projects are durable, individuals risk their utility losses accumulating over time if they repeatedly belong to the minority under the simple majority rule. If individuals are risk-averse, this is undesirable from an ex ante perspective. In this paper we propose minority voting as a scheme that can partially protect individuals from the risk of repeated exploitation. Minority voting is most conveniently demonstrated in a two-period set-up. Under minority voting, only individuals belonging to the minority in the first period are allowed to vote in the second period.

The core idea and motivation of minority voting in the context of social decisions can be illustrated by a simple example. Three persons who enjoy going out together can either go to the cinema, where they can choose between two films, or to a restaurant, where they again have a choice of two, e.g. Italian or Chinese. Suppose that, using simple majority rule, two of them can assert their preferred decision in the first stage (e.g. going to the cinema), while the third person might have preferred the other alternative (going to a restaurant). In order to minimize the losses that occur by going to the cinema, the third person, i.e. the minority in the first stage, obtains the sole right to choose which film they will see. This example is a demonstration of minority voting. A group takes several decisions in series, and these decisions are linked because the second choice depends on the first. The minority in the first instance is given the exclusive right to make the choice in the second stage.

The same idea is illustrated by the following infrastructure project example where a community decides about two technologically independent projects. Suppose a city making a decision first on whether to build a new expressway and second on whether to build an airport, thus increasing air traffic and noise. The city can decide on each project by simple majority voting. Under minority voting, the expressway is also decided upon by the simple majority rule. However, only the minority from the first stage of voting will vote on whether to build the airport. Such a procedure gives individuals who may be living close to the new road and thus suffering from increasing noise a better chance of protecting themselves from further noise pollution due to additional air traffic. Vice versa, individuals who want better and faster traffic routes may have better chances of realizing at least one of the infrastructure projects, as a defeat in the first stage may exclude opponents from involvement in the decision on the airport.

Our examples illustrate the basic trade offs of minority voting. In comparison with repeated simple majority rule, minority voting, on the one hand, protects individuals from being outvoted repeatedly. On the other hand, some individuals are excluded from decisions in the second period, which creates negative externalities. The theme of this paper is to identify the circumstances under which minority voting is preferable to repeated simple majority voting for technologically independent projects. Our findings indicate that minority voting is superior to repeated simple majority voting as soon as the degree of risk aversion exceeds some threshold value. In such circumstances, increasing the future voting power of minorities today goes hand in hand with an increase in aggregate efficiency.

Minority voting may invite strategic behaviour. Under minority voting, the property of all equilibria in the first period is the formation of the smallest majority and thus largest minority as being in the majority eliminates the future voting right. Hence, individuals will only support a project in the first period if their benefits are high and if they are pivotal in forming a majority. Since the minority in the first period is always composed of almost half of the committee, the danger of dictatorship in future decisions is avoided by strategic voting.[1]

Our paper is part of the literature on linking voting across problems. Jackson and Sonnenschein (2005) show that, when problems are repeated many times, full efficiency can be reached at the limit and that this insight essentially applies to any collective decision problem. Casella (2005) introduces storable votes mechanisms, where a committee makes binary decisions repeatedly over time and where agents may store votes over time.[2] Qualitative voting as introduced by Hortala-Vallve (2005) is closely related to storable votes. Individuals obtain a stock of votes at the start and hence can decide in the following ballots on what issue to exert more or less influence, i.e. to cast more

[1]For a committee of three members the minority necessarily consists of one member, hence the decision scheme is dictatorial in the second period.

[2]Cummulative voting is closely related to the storable votes mechanism as under this voting scheme individuals can again cast more than one vote for one alternative (see for example Sawyer and MacRae (1962), Brams (1975), Cox (1990), Guinier (1994) or Gerber, Morton, and Rietz (1998)).

or less votes.

Linkages of voting across issues can also occur through vote trading which goes back at least to Buchanan and Tullock (1962) and Coleman (1966), and has been developed amongst others by Brams and Riker (1973), Ferejohn (1974), Philipson and Snyder (1996) or Piketty (1994). We focus on other collective decision problems, where earlier decisions may have a durable impact, thus making it very costly to be in the minority at the earlier stage and very valuable to have more voting rights in the future. By introducing minority voting, we make a new proposal for addressing the most difficult issue in designing collective decisions: how to design processes that bridge two core principles: (a) 'same right for each person to influence outcomes' and (b) 'respecting and protecting minorities' (see e.g. Guinier (1994) or Issacharoff, Karlan, and Pildes (2002)).

The paper is structured as follows: In section 2 we introduce the model, e.g. the society, the utility functions and the new voting scheme. The voting equilibria of the two-period game of our model are described in section 3. We provide a general welfare comparison in section 4. Section 4.2 deals with case where the first-period project is not adopted and hence no utility impact of the first project on the second period occurs. The case where the first-period project is realized will be discussed in section 4.3. The last part of the interim comparison is given in section 4.4 where we assume that individuals have constant absolute risk aversion, i.e. we derive a result for a special utility function. An ex-ante comparison between both voting schemes in the case of constant absolute risk-aversion is presented in section 5. This leads to the main result of this paper. In section 7 we compare MV with SM regarding the axiomatic properties. Section 8 concludes, and the Appendix contains the proofs (part A), tables (part B) and graphics (part C).

2 The Model

2.1 The Set-up

We consider a committee of N (N uneven) individuals that meets twice – today (period $t = 1$) and tomorrow (period $t = 2$) – to vote on a proposal that affects the utility of all members. In each period $t = 1, 2$, the collective decision is binary. Specifically, the committee decides in each period between the status quo and a project.

2.2 Utility Functions and Aggregate Welfare

The payoff in both periods is given as follows:

- In the first period, individual i's utility is given by $u_{i1} = f(a_1 z_{i1})$. The committee decision is represented by the indicator variable a_1 and assumes two values: $a_1 = 0$ denotes the status quo, and $a_1 = 1$ represents the new project, which changes the status quo. The variable z_{i1} represents the benefits from the project for individual i in period $t = 1$. We assume that z_{i1} takes one of the values $\{-1, -\gamma, \gamma, 1\}$, $\gamma \in (0, 1)$, with equal probability. The utility function f is assumed to be concave, strictly increasing and integrable. The utility of the status quo is normalized to zero. Individuals with $z_{i1} \geq 0$ ($z_{i1} < 0$) are called project winners (project losers) respectively. Individuals are ordered by their first-period preferences, i.e. $i < j \Rightarrow z_{i1} \leq z_{j1}$ has to hold.

- We assume that the project in the first period may be durable, i.e. it has lasting consequences, and it represents the status quo in the second period if $a_1 = 1$. In the second period, a new project arises upon which the committee decides. We use a_2 to denote the indicator variable specifying whether the new project in period $t = 2$ is adopted ($a_2 = 1$) or rejected ($a_2 = 0$). Such a set-up is in line with the spirit of our infrastructure example in the Introduction. Utility of individual i in the second period is given by

$$u_{i2} = f(\epsilon a_1 z_{i1} + a_2 z_{i2})$$

where z_{i2} is assumed to be uniformly distributed on $[-1, 1]$ and ϵ is the depreciation rate of the impact of the first-period project. The polar cases $\epsilon = 1$ ($\epsilon = 0$)

cover the cases where the first-period project is completely (not) durable.

- Overall expected utility of individual i is given by

$$\begin{aligned} W &:= W_1 + \delta W_2 \\ \text{where } W_1 &:= \sum_{i=1}^{N} u_{i1} \text{ is the first-period aggregate utility,} \\ W_2 &:= \sum_{i=1}^{N} u_{i2} \text{ is the second-period aggregate utility,} \\ \delta &\in (0,1] \text{ denotes the discount factor.} \end{aligned}$$

We will do an interim comparison in section 4, i.e. in the case where first-period preferences $\{z_{i1}\}_{i=1}^{N}$ are given, and an ex ante comparison in section 5 where the $\{z_{i1}\}_{i=1}^{N}$ are not yet distributed. Hence, we define

$$\begin{aligned} \mathbb{E}_1 &:= \text{ expected value in the ex ante case, i.e. at the beginning of } t = 1 \\ \mathbb{E}_2 &:= \text{ expected value in the interim case, i.e. at the beginning of } t = 2 \end{aligned}$$

We add the index MV (minority voting) or SM (simple majority rule) to W_1 and W_2 to distinguish between the two voting schemes (the corresponding definitions are presented in the next section). We use the utilitarian criterion, i.e. the minority voting scheme MV is called *superior to* or *socially more efficient than* SM if $W^{MV} \geq W^{SM}$. Furthermore, we assume that the committee members observe the realization of utilities in the first period so that the values $\{z_{i1}\}_{i=1}^{N}$ are common knowledge.

Some remarks regarding our set-up are in order. We work with discrete uniform distribution in the first period, as this simplifies the formal analysis and makes it tractable. While we could also work with a discrete distribution in the second period, a continuous distribution allows an easier presentation of the proofs. The assumption that first-period utilities are commonly known is essential, and we comment on the imcomplete information case in the concluding section.

2.3 Voting Rules

We compare two voting schemes: repeated simple majority voting (SM) and minority voting (MV). While SM is standard and the equilibria in a setting with only two alternatives are obvious, MV is a new voting scheme defined as follows: In the first period, a simple open majority voting scheme takes place. If the status quo and the project receive the same number of votes, the winner is determined by a tie-breaking procedure where status quo and the project are selected with probability $\frac{1}{2}$. Since voting takes place openly, we can split the committee into three groups: voting winners, voting losers, and absentees, depending on whether an individual has voted for the winning alternative, the losing alternative or has abstained from voting. The number of voting winners, voting losers, and absentees is denoted by N^w, N^l and N^a, respectively. Thus, we have

$$N = N^w + N^l + N^a, \;\; N^w \geq N^l$$

Minority voting is now simply defined as

MV: *In the second period, only voting losers are allowed to vote.*[3]

Note that voting losers are not necessarily project losers, i.e. individuals with negative utility, if the first-period project is realised. Similarly, voting winners may not be project winners.

It is obvious that MV invites strategic voting. Suppose that in the first period members vote sincerely. Suppose that the size of the majority is larger than $\frac{N+1}{2}$. Then, at least one member of the majority has an incentive to vote against his preferences. The outcome is not affected but this member can preserve his voting right for the second period. However, such attempts are limited by the fact that joining voting losers may turn this group into voting winners, thus eliminating future voting rights.

[3] A variant of MV is to allow absentees in the first period to keep their voting right. Then, only voting losers and absentees are allowed to vote in the second period.

3 Equilibria under MV

In this section we analyze the existence and style of voting equilibria under MV. In order to eliminate implausible equilibria, we exclude weakly dominated voting strategies and look for perfect Bayesian Nash equilibria in pure strategies. We start with the second period where the simple majority rule is applied. Therefore, all individuals i who have voting right in $t = 2$ will vote in favor of their preferences z_{i2} (if weakly dominated strategies are eliminated). This uniquely determines the second-period equilibrium. We will therefore concentrate next on the first period.

3.1 Existence

We start with some simple observations regarding the nature of perfect Bayesian Nash equilibria under the MV rule. Let I^- be the set of all project losers in the first period:

$$I^- = \{i \in \{1, .., N\} : \; z_{i1} < 0\}$$

The number of project losers will be denoted by k, i.e. $k = |I^-|$.[4]

Proposition 1

For first-period equilibria the following statement holds:

(i) $N^a = 0$.

(ii) If the number of project losers $k \geq \frac{N+1}{2}$*, then* $\frac{N+1}{2}$ *project losers will vote for* $a_1 = 0$*. The rest of the committee favors* $a_1 = 1$*. The status quo prevails.*

(iii) If $k < \frac{N+1}{2}$*, then* $\frac{N+1}{2}$ *project winners will vote for* $a_1 = 1$*. All other individuals will vote for* $a_1 = 0$*. The project is adopted.*

The proof of Proposition 1 is given in Appendix A.

We note that the equilibrium is not unique. Suppose there are $k < \frac{N+1}{2}$ project losers, then we have $\binom{N-k}{\frac{N-1}{2}-k} = \binom{N-k}{\frac{N+1}{2}}$ possible partitions of the project winners into two groups, i.e. $\frac{N+1}{2}$ individuals who vote for the project and belong to the voting winners, and $\frac{N-1}{2} - k$ individuals who vote against it and thereby belong to the voting losers.

[4] $|S|$ denotes the number of pairwise different elements in a set S.

Any combination of strategies that satisfies Proposition 1 represents an equilibrium. This is summarized by the following corollary:

Corollary 1

In the first period there are $\binom{N-k}{\frac{N+1}{2}}$ *voting equilibria under MV if* $k < \frac{N+1}{2}$*, and* $\binom{k}{\frac{N+1}{2}}$ *voting equilibria if* $k \geq \frac{N+1}{2}$.

3.2 Equilibrium Refinement

The decision of individual i in $t = 1$ will be denoted by a_{i1}, i.e. $a_{i1} = 1$ if individual i votes for the change or project and $a_{i1} = 0$ if individual i votes against it. An equilibrium is given by a tuple of decision strategies $(a_{i1})_{i\in\{1,\ldots,N\}}$. As a consequence of the indeterminacy of equilibria in $t = 1$, we apply a plausible refinement that we will call 'Maximal Magnanimity' (MM).

MM: *Voters who benefit most from a decision form the majority, i.e.*

- *if* $k \geq \frac{N+1}{2}$, then $a_{i1} = \begin{cases} 0, & i \leq \frac{N+1}{2} \\ 1, & \text{otherwise} \end{cases}$
- *if* $k < \frac{N+1}{2}$, then $a_{i1} = \begin{cases} 0, & i < \frac{N+1}{2} \\ 1, & \text{otherwise} \end{cases}$.

MM is a coordination device with the idea that individuals who benefit more from a decision than others do not take advantage of joining the minority (recall that we assumed in section 2.2 that individuals are ordered by their first-period benefits z_{i1}). Individuals who benefit little may switch sides in order to keep their voting right, if this does not affect the outcome.

As a tie-breaking rule, we assume that in the case of indifference individuals have the same probability of belonging to the minority: Suppose we have a committee consisting of 21 members of whom 5 members have first-period benefits of -1 (they obtain indices between 1 and 5), 3 members with first-period benefits of $-\gamma$ (they obtain the indices 6 - 8 and vote against the project), and 13 members with benefits of 1. The last group divides itself into voting winners and voting losers by fair randomization to obtain the equilibrium voting behaviour, i.e. the members draw their own index (between 9 and 21) randomly. The resulting voting strategy is a pure strategy according to MM.

3.3 Welfare Comparison

The following lemma simplifies the welfare comparison.

Lemma 1

MV is interim superior to SM if and only if $W_2^{MV} \geq W_2^{SM}$.

Lemma 1 follows from the observation that the decision in the first period is the same under MV and SM, i.e.

$$a_1 = 1 \Leftrightarrow |\{i : z_{i1} \geq 0\}| \geq \frac{N+1}{2}.$$

4 The Second Period

In this section we compare $\mathbb{E}_2[W_2^{MV}]$ and $\mathbb{E}_2[W_2^{SM}]$ and derive some general expressions for the welfare comparison between SM and MV in the second period.

4.1 General Characterization

In the second period, voting equilibria are unique and individuals vote sincerely. Every individual who has the right to vote will select $a_{i2} = 1$ if and only if $z_{i2} \geq 0$ as this is a strictly dominant strategy for individuals with $z_{i2} > 0$ and we have assumed the tie-breaking rule that individuals who are indifferent vote in favor of the project. The voting schemes only differ with respect to the number of individuals participating in the tally. As stated in Proposition 1, the equilibrium number of individuals who keep their voting right in the second period under MV is $\frac{N-1}{2}$. While the whole committee votes under SM, only $\frac{N-1}{2}$ individuals vote under MV.

Suppose in $t = 2$ there are w individuals who are allowed to vote. The expected welfare of an agent who has certain preferences in $t = 2$ depends on the probability of his belonging to the winning majority, denoted by $P(w)$. If w is odd, this occurs if at least $\frac{w-1}{2}$ persons vote in the same way. If w is even, this occurs either if $\frac{w}{2}$ individuals vote in the same way or there is a tie-break in favor of the preferences of the agent under consideration in the case of a tally, i.e. if $\frac{w}{2}-1$ persons vote in the same way. $P(w)$ is given by

$$P(w) = \begin{cases} \frac{1}{2^{(w-1)}} \sum_{i=\frac{w-1}{2}}^{w-1} \binom{w-1}{i}, & \text{if } w \text{ odd} \\ \frac{1}{2^{(w-1)}} \sum_{i=\frac{w}{2}}^{w-1} \binom{w-1}{i} + \frac{1}{2}\binom{w-1}{\frac{w}{2}-1}\frac{1}{2^{w-1}}, & \text{if } w \text{ even} \end{cases}$$

The second term of $P(w)$ in case w is even stems from the tie-breaking rule. By using the following equalities we can simplify the former expression:

$$\begin{aligned} \lfloor x \rfloor &:= \max\{n \in \mathbb{N} : n \leq x\} & \\ 2^{w-1} &= \textstyle\sum_{i=0}^{w-1} \binom{w-1}{i} & \\ \Rightarrow \quad \textstyle\sum_{i=\frac{w-1}{2}}^{w-1} \binom{w-1}{i} &= 2^{w-2} + \frac{1}{2}\binom{w-1}{\frac{w-1}{2}} & \text{if } w \text{ odd} \\ \textstyle\sum_{i=\frac{w}{2}}^{w-1} \binom{w-1}{i} &= 2^{w-2} & \text{if } w \text{ even} \\ \Rightarrow \quad P(w) &= \frac{1}{2^w}\binom{w-1}{\lfloor\frac{w-1}{2}\rfloor} + \frac{1}{2} & \end{aligned}$$

As the number of individuals with a voting right in $t = 2$ equals N under SM, while only $\frac{N-1}{2}$ individuals vote under MV, the expected utility of a first-period voting loser under MV is higher than under SM: $P(N) < P(\frac{N-1}{2})$. The probability of winning decreases if the number of individuals with voting right increases. By contrast, voting winners in $t = 1$ under MV have no influence on the decision in $t = 2$. The probability that their preferred decision will be taken is equal to $\frac{1}{2}$, which is smaller than $P(N)$. Hence voting winners from $t = 1$ are worse off under MV than voters under SM. We will show that if risk aversion exceeds some threshold the gain of being a voting loser under MV outweighs ex ante the loss of being a voting winner. In the following we define some general formula for the aggregate welfare.

4.1.1 Simple Majority Rule

Given a vector of realizations $\{z_{i1}\}_{i=1}^{N}$ the second-period aggregate expected utility is given by

$$
\begin{aligned}
\mathbb{E}_2[W_2^{SM}] &= \sum_{i=1}^{N}\Big[\frac{1}{2}P(N)\int_0^1 f(\epsilon a_1 z_{i1}+z_{i2})dz_{i2}+\frac{1}{2}(1-P(N))f(\epsilon a_1 z_{i1}) \\
&\quad +\frac{1}{2}P(N)f(\epsilon a_1 z_{i1})+\frac{1}{2}(1-P(N))\int_{-1}^{0} f(\epsilon a_1 z_{i1}+z_{i2})dz_{i2}\Big] \\
&= \frac{1}{2}\sum_{i=1}^{N}\Big[P(N)\int_0^1 f(\epsilon a_1 z_{i1}+z_{i2})dz_{i2} \\
&\quad +(1-P(N))\int_{-1}^{0} f(\epsilon a_1 z_{i1}+z_{i2})dz_{i2}+f(\epsilon a_1 z_{i1})\Big].
\end{aligned}
$$

The aggregate expected utility mainly consists of four parts for each individual i: Having non-negative second-period benefit z_{i2} and winning with probability $\frac{1}{2}P(N)$ (hence, $a_2 = 1$), having non-negative z_{i2} but losing (i.e. $a_2 = 0$) with probability $\frac{1}{2}(1-P(N))$, having negative z_{i2} and winning (i.e. $a_2 = 0$) with probability $\frac{1}{2}P(N)$, and finally having negative z_{i2} and losing (i.e. $a_2 = 1$) with probability $\frac{1}{2}(1-P(N))$. This basic structure is also given under MV, but there we have to distinguish between the first $\frac{N-1}{2}$ individuals who keep the voting right and hence win with probability $P(\frac{N-1}{2})$ and the last $\frac{N+1}{2}$ individuals who loose their voting right and win in the second period with probability $\frac{1}{2}$. This calculation is given in the next section.

It will be useful to define

$$
\begin{aligned}
\Delta\mathbb{E}_2[W_2^{SM}] &= 2\mathbb{E}_2[W_2^{SM}]-\sum_{i=1}^{N} f(\epsilon a_1 z_{i1}) \\
&= \sum_{i=1}^{N}\Big[P(N)\int_0^1 f(\epsilon a_1 z_{i1}+z_{i2})dz_{i2}+(1-P(N))\int_{-1}^{0} f(\epsilon a_1 z_{i1}+z_{i2})dz_{i2}\Big].
\end{aligned}
$$

$\Delta\mathbb{E}_2[W_2^{SM}]$ captures the differential impact on aggregate expected utility if the second project is undertaken in $t = 2$. It will be easier to analyze than $\mathbb{E}_2[W_2^{SM}]$.

4.1.2 Minority Voting

As we have shown in Proposition 1, the number of voting losers in equilibrium equals $\frac{N-1}{2}$ for N odd. Because of MM, the first $\frac{N-1}{2}$ individuals form the minority if $a_1 = 1$. If $a_1 = 0$ the minority is formed by individuals $\frac{N+3}{2}..N$. In that case $\mathbb{E}_2[u_{i2}] = \mathbb{E}_2[\overline{u}]$ for all individuals i, since no utility impact occurs from the first-period project. Therefore a change in the numeration of the individuals does not affect the value $\mathbb{E}_2[W_2^{MV}]$: without loss of generality we can assume that individuals $1...\frac{N-1}{2}$ form the minority. The aggregate expected utility under minority voting in $t = 2$ is given by

$$\begin{aligned}
\mathbb{E}_2[W_2^{MV}] &= \sum_{i=1}^{\frac{N-1}{2}} \Big[\frac{1}{2}P(\frac{N-1}{2})\int_0^1 f(\epsilon a_1 z_{i1} + z_{i2})dz_{i2} + \frac{1}{2}(1-P(\frac{N-1}{2}))f(\epsilon a_1 z_{i1}) \\
&\quad + \frac{1}{2}(1-P(\frac{N-1}{2}))\int_{-1}^0 f(\epsilon a_1 z_{i1} + z_{i2})dz_{i2} + \frac{1}{2}P(\frac{N-1}{2})f(\epsilon a_1 z_{i1})\Big] \\
&+ \sum_{i=\frac{N+1}{2}}^{N} \Big[\frac{1}{4}\int_0^1 f(\epsilon a_1 z_{i1} + z_{i2})dz_{i2} + \frac{1}{4}f(\epsilon a_1 z_{i1}) \\
&\quad + \frac{1}{4}\int_{-1}^0 f(\epsilon a_1 z_{i1} + z_{i2})dz_{i2} + \frac{1}{4}f(\epsilon a_1 z_{i1})\Big] \\
&= \frac{1}{2}\Big[\sum_{i=1}^{\frac{N-1}{2}}\Big(P(\frac{N-1}{2})\int_0^1 f(\epsilon a_1 z_{i1} + z_{i2})dz_{i2} + (1-P(\frac{N-1}{2}))\int_{-1}^0 f(\epsilon a_1 z_{i1} + z_{i2})dz_{i2}\Big) \\
&+ \frac{1}{2}\sum_{i=\frac{N+1}{2}}^{N}\int_{-1}^1 f(\epsilon a_1 z_{i1} + z_{i2})dz_{i2}\Big] + \frac{1}{2}\sum_{i=1}^{N} f(\epsilon a_1 z_{i1})
\end{aligned}$$

Again we define:

$$\begin{aligned}
\Delta\mathbb{E}_2[W_2^{MV}] &= 2\mathbb{E}_2[W_2^{MV}] - \textstyle\sum_{i=1}^N f(\epsilon a_1 z_{i1}) \\
&= \textstyle\sum_{i=1}^{\frac{N-1}{2}}\Big[P(\frac{N-1}{2})\int_0^1 f(\epsilon a_1 z_{i1} + z_{i2})dz_{i2} + (1-P(\frac{N-1}{2}))\int_{-1}^0 f(\epsilon a_1 z_{i1} + z_{i2})dz_{i2}\Big] \\
&\quad + \tfrac{1}{2}\textstyle\sum_{i=\frac{N+1}{2}}^{N}\int_{-1}^1 f(\epsilon a_1 z_{i1} + z_{i2})dz_{i2}
\end{aligned}$$

4.1.3 Comparison

As stated in Lemma 1 we need to derive a condition for $\mathbb{E}_2[W_2^{MV}] \geq \mathbb{E}_2[W_2^{SM}]$ to show under what circumstances MV is superior to SM. We will analyze the general case where second-period utility is given by a concave and increasing utility function $f(\epsilon a_1 z_{i1} + z_{i2})$. We will use the following lemma for the upcoming analysis:

Lemma 2

$\mathbb{E}_2[W_2^{MV}] \geq \mathbb{E}_2[W_2^{SM}]$ *is equivalent to* $\Delta\mathbb{E}_2[W_2^{MV}] \geq \Delta\mathbb{E}_2[W_2^{SM}]$.

Note that the expression $a_1 z_{i1}$ is either $z_{i1} \in \{-1, -\gamma, \gamma, 1\}$ in the case where the first project is adopted ($a_1 = 1$) or zero if the project is rejected ($a_1 = 0$). To simplify notation, in the following we use z_{i1} instead of $a_1 z_{i1}$ which remains unaffected in the case of adoption ($a_1 = 1$) and which is set to zero in the case of rejection ($a_1 = 0$).

Futhermore, we define

$$F(z_{i1}, z_{i2}) := \int_{-1}^{z_{i2}} f(\epsilon z_{i1} + \tilde{z}_{i2}) d\tilde{z}_{i2}.$$

With this notation we obtain for the second period

Proposition 2

The MV voting scheme is superior to the SM rule if and only if

$$\begin{aligned}(P(\frac{N-1}{2}) - P(N)) \sum_{i=1}^{\frac{N-1}{2}} \Big[F(z_{i1}, 1) \quad &- \quad 2F(z_{i1}, 0) + F(z_{i1}, -1)\Big] \\ -(P(N) - \frac{1}{2}) \sum_{i=\frac{N+1}{2}}^{N} \Big[F(z_{i1}, 1) \quad &- \quad 2F(z_{i1}, 0) + F(z_{i1}, -1)\Big] \geq 0 \qquad (1)\end{aligned}$$

The proof of Proposition 2 is given in Appendix A.

Let $A(z_{i1})$ denote the term $F(z_{i1}, 1) - 2F(z_{i1}, 0) + F(z_{i1}, -1)$. With this notation we obtain

Lemma 3

Suppose f is a concave, increasing function. Then

(i) $A(z_{i1}) \geq 0$ *for all* $i \in \{1, ..., N\}$.

(ii) $A(z_{i1}) \geq A(z_{j1}) \Leftrightarrow z_{i1} \leq z_{j1}$

The proof of Lemma 3 is given in Appendix A.

The condition for MV to be superior to SM stated in Proposition 2 amounts to

$$(P(\frac{N-1}{2})-P(N))\sum_{i=1}^{\frac{N-1}{2}}A(z_{i1})-(P(N)-\frac{1}{2})\sum_{i=\frac{N+1}{2}}^{N}A(z_{i1})\geq 0$$

Recall that $P(\frac{N-1}{2}) > P(N) > \frac{1}{2}$. Hence, the first summand represents the expected utility gain of voting losers if MV is used. The second summand indicates the utility loss of first-period voting winners. Their probability of winning shrinks in the second period. Hence, the question we have to answer is whether the gain of voting losers can outweigh the expected utility loss of voting winners.

The overall comparison MV versus SM depends crucially on two parameters. First, the more durable a project is, i.e. the higher ϵ is, the lower is the loss of first-period voting winners under MV. Their utility gain from the first period lessens potential loss in the second period. Second, if people are more risk-averse, they will want to reduce the risk of large losses, so the MV voting scheme becomes more attractive to them.

4.2 The case $a_1 = 0$

In this section we analyze the relation between SM and MV if the status quo prevails in $t = 1$, i.e. if $a_1 = 0$.[5] In that case the utility of the second period is independent of z_{i1}, i.e. $A(z_{i1}) = A\,\forall\, i \in \{1, ..., N\}$. Hence, condition (1) in Proposition 2 is equivalent to

$$A\cdot\Big(\frac{N-1}{2}[P(\frac{N-1}{2})-P(N)]-\frac{N+1}{2}[P(N)-\frac{1}{2}]\Big)\geq 0 \tag{2}$$

If we assume strictly increasing utility functions f we have $A > 0$. Thus inequality (2) is equivalent to

$$\frac{N-1}{2}[P(\frac{N-1}{2})-P(N)]-\frac{N+1}{2}[P(N)-\frac{1}{2}]\geq 0 \tag{3}$$

With this inequality we can show the following proposition.

Proposition 3

Suppose a committee of N members whose utility can be described by a strictly increasing, concave function f. In the case where $a_1 = 0$ (i.e. $z_{1,1}, ..., z_{\frac{N+1}{2},1} < 0$), SM is superior to MV.

[5]The rejection of a project is equivalent to the case where projects are not durable, i.e. $\epsilon = 0$.

The proof can be found in Appendix A.

The intuition for Proposition 3 is obvious. If $a_1 = 0$, there is no impact from first-period choices on second-period utility. Hence, there is no risk of repeated exploitation and hence, restricting voting rights is not socially desirable.

Note that we obtain the same result if $\epsilon = 0$, i.e. if the first project (realized or not) exhibits no utility impact on the second period.

4.3 Durable Projects: $a_1 = 1$ and $\epsilon \neq 0$

In this section we analyze the case where the first-period project is accepted. Individuals who suffer from this project get a higher probability of compensation for this utility loss under MV. We proceed with the result of Proposition 2, i.e. inequality (1) in general.

Proposition 2 states that MV is superior to SM if and only if inequality (1) is fulfilled. If the project of period 1 is accepted and $\epsilon > 0$, then all individuals with different z_{i1}-values have different expected utilities for $t = 2$, i.e. we cannot neglect the index i. Given this situation, we still can simplify inequality (1) by using the median value of the $A(z_{i1})$'s. We define $A_L := \frac{2}{N-1}\sum_{i=1}^{\frac{N-1}{2}} A(z_{i1})$ as the average $A(z_{i1})$ value of the first-period voting losers and $A_W := \frac{2}{N+1}\sum_{i=\frac{N+1}{2}}^{N} A(z_{i1})$ respectively for the voting winners. With these definitions inequality (1) is equivalent to

$$
\begin{aligned}
\sum_{i=1}^{\frac{N-1}{2}}[P(\tfrac{N-1}{2}) - P(N)]A_L - \sum_{i=\frac{N+1}{2}}^{N}[P(N) - \tfrac{1}{2}]A_W &\geq 0 \\
\Leftrightarrow \qquad \tfrac{N-1}{2}[P(\tfrac{N-1}{2}) - P(N)]A_L &\geq \tfrac{N+1}{2}[P(N) - \tfrac{1}{2}]A_W \\
\Leftrightarrow \qquad \frac{(N-1)[P(\frac{N-1}{2})-P(N)]}{(N+1)[P(N)-\frac{1}{2}]} &\geq \frac{A_W}{A_L}
\end{aligned}
\tag{4}
$$

- The LHS of inequality (4) $\frac{(N-1)[P(\frac{N-1}{2})-P(N)]}{(N+1)[P(N)-\frac{1}{2}]} =: Q(N)$ is a fixed value for N given. We have shown in the proof of Proposition 3 that $Q(N) < 1 \; \forall \; N \in \mathbb{N}$, N odd.

- The RHS depends on several parameters, e.g. the first-period benefits z_{i1} of all individuals or other parameters that might be needed for the utility function f, e.g. a level of risk aversion. We already know from Lemma 3 that $A_W \leq A_L$, i.e. the RHS is also less than or equal to 1.

In the following we derive an approximation for the LHS. The RHS depends on the utility function f which is not specified yet. Therefore, we will derive more results for the RHS in the next section where we assume a specific utility function that exhibits constant absolute risk aversion. We will show that the RHS is decreasing in the level of risk aversion and hence, that there exists a level of risk aversion r such that inequality (4) is fulfilled, i.e. MV is superior to SM.

An Approximation for the LHS

The function $P(w)$ describes the value of a binomially distributed random variable X: $P(w) = \mathbb{P}(X \geq \frac{w-1}{2})$, i.e. the probability that at least $\frac{w-1}{2}$ other individuals vote in the same way as the individual under consideration. Binomially distributed random variables can be approximated by the normal distribution (for further details see Appendix A).

We obtain

$$P(\frac{N-1}{2}) \sim 1 - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\lfloor \frac{N-3}{4} \rfloor^{-\frac{1}{2}}} \exp(-y^2)dy$$

and

$$P(N) \sim 1 - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{(\frac{N-1}{2})^{-\frac{1}{2}}} \exp(-y^2)dy$$

With this approximation we can calculate the limit of the LHS. [6]

$$\lim_{N\to\infty} (\frac{(N-1)(P(\frac{N-1}{2}) - P(N))}{(N+1)(P(N) - \frac{1}{2})}) = \lim_{N\to\infty} Q(N) = \sqrt{2} - 1$$

The following graph shows $Q(N)$ for $N \in [3, 150]$. The horizontal line indicates the limit $\sqrt{2} - 1$. Note that $Q(N)$ is always greater than $\sqrt{2} - 1$ for all N that can be

[6] The whole calculation is given in Appendix A.

described by $N = 4b+3$ with $b \in \mathbb{N}$ (in the following we will denote this by $N \equiv 3(4)$), while $Q(N) < \sqrt{2} - 1$ if $N \equiv 1(4)$. The reason is that the size of the minority in equilibrium is an even number if $N \equiv 1(4)$ and we have $P(2b) = P(2b+1) \, \forall \, b \in \mathbb{N}$: The probability of winning in $t = 2$ - and hence the probability of minimizing loss in the first period - does not increase monotonically in $\frac{N-1}{2}$ under MV ($\frac{N-1}{2}$ can take odd and even numbers), while the probability does increase monotonically in N under SM (we consider only odd numbers of individuals). Comparing MV and SM, the probability of winning in $t = 2$ - and hence the probability of minimizing loss in the first period - is lower if $N \equiv 1(4)$ and higher if $N \equiv 3(4)$.

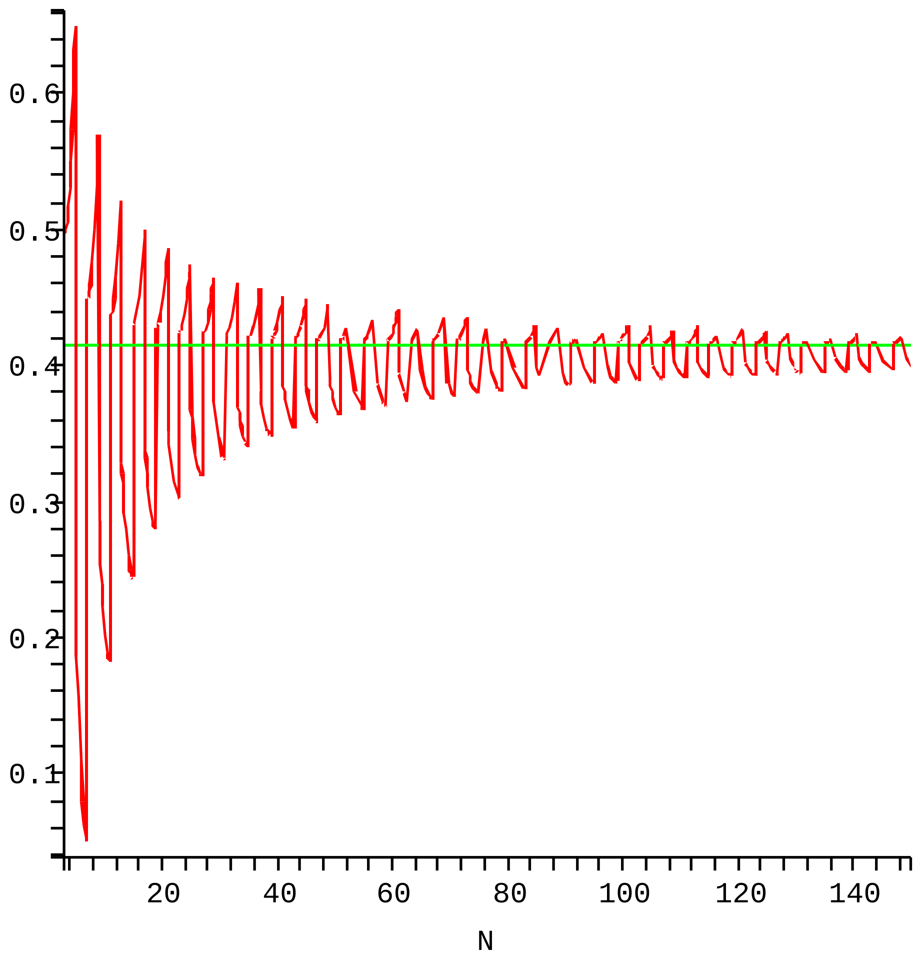

Figure 1: $Q(N) = \frac{(N-1)(P(\frac{N-1}{2})-P(N))}{(N+1)(P(N)-\frac{1}{2})}$

The function $Q(N)$ is oscillating, which makes it difficult to derive a general condition on $\frac{A_W}{A_L}$. We therefore use the limit of $Q(N)$ just derived as a lower boundary of $Q(N)$ if $N \equiv 3(4)$ and obtain the condition: MV is superior to SM if

$$\sqrt{2} - 1 \geq \frac{A_W}{A_L}.$$

This condition is only sufficient if $\frac{N-3}{4} \in \mathbb{N}$. An immediate consequence is that we have to differentiate between cases where either $N \equiv 3(4)$ or $N \equiv 1(4)$.

4.4 Utility with Constant Absolute Risk Aversion

In this section we apply the result derived in section 4.3 on the concave utility function

$$f(z_{i1}, z_{i2}) = -\exp(-r(\epsilon a_1 z_{i1} + z_{i2})).$$

The parameter $r > 0$ can be used to describe the risk aversion of the individuals. This function exhibits constant absolute risk aversion.[7] No matter how large or small the utility, or rather the potential utility loss or gain is, risk aversion always remains constant.

Benefits z_{i1} are already known. Again we assume throughout this section that $a_1 = 1$ (the case $a_1 = 0$ was given in section 4.2) and $\epsilon > 0$, i.e. the first project is durable. Hence, more than half of the individuals have a non-negative utility gain from the project proposed in period 1.

The condition we have to check is condition 4: MV is superior to SM if

$$\frac{(N-1)[P(\frac{N-1}{2}) - P(N)]}{(N+1)[P(N) - \frac{1}{2}]} \geq \frac{A_W}{A_L}.$$

We have shown that the LHS converges to $\sqrt{2} - 1$ as N becomes large. With the concrete utility function $f(z_{i1}, z_{i2}) = -e^{-r(\epsilon z_{i1} - z_{i2})}$ we can now analyze the RHS in more detail.

As the critical function $A(z_{i1})$ we obtain

$$A(z_{i1}) = \frac{1}{r} \exp(-r\epsilon z_{i1}) \Big[\exp(-r) + \exp(r) - 2 \Big]$$

[7]Absolute risk aversion is given by $-\frac{f''(x)}{f'(x)} = r$ with $x = \epsilon a_1 z_{i1} + z_{i2}$..

From this it follows that

$$\begin{aligned}
\frac{A_W}{A_L} &= \frac{\frac{2}{N+1}\sum_{j=\frac{N+1}{2}}^{N} A(z_{j1})}{\frac{2}{N-1}\sum_{i=1}^{\frac{N-1}{2}} A(z_{i1})} = \frac{\frac{2}{N+1}\sum_{j=\frac{N+1}{2}}^{N} \exp(-r\epsilon z_{j1})}{\frac{2}{N-1}\sum_{i=1}^{\frac{N-1}{2}} \exp(-r\epsilon z_{i1})} \\
&= \frac{N-1}{N+1}\,\frac{\sum_{j=\frac{N+1}{2}}^{N} \exp(-r\epsilon z_{j1})}{\sum_{i=1}^{\frac{N-1}{2}} \exp(-r\epsilon z_{i1})}
\end{aligned}$$

where $\exp(-r\epsilon z_{i1}) \geq \exp(-r\epsilon z_{j1})\ \forall j \geq i$. Therefore we can conclude that if there exist $i, j \in \{1, ..., N\}$ with $z_{i1} \neq z_{j1}$ then $A_W < A_L$, i.e. $\frac{A_W}{A_L}$ is strictly smaller than 1.

We want to show that there exists a $r \geq 0$ such that for all combinations of first-period benefits $\{z_{i1}\}_{i\in\{1,\ldots,N\}}$ MV is better than SM. Therefore the next question is under what conditions $\frac{A_W}{A_L}$ is monotonically decreasing in r.

$$\begin{aligned}
0 &\geq \frac{\partial \frac{A_W}{A_L}}{\partial r} \\
\Leftrightarrow\quad 0 &\geq \sum_{j=\frac{N+1}{2}}^{N} (-\epsilon z_{j1}) \exp(-r\epsilon z_{j1}) \sum_{i=1}^{\frac{N-1}{2}} \exp(-r\epsilon z_{i1}) \\
&\quad - \sum_{i=1}^{\frac{N-1}{2}} (-\epsilon z_{i1}) \exp(-r\epsilon z_{i1}) \sum_{j=\frac{N+1}{2}}^{N} \exp(-r\epsilon z_{j1}) \\
\Leftrightarrow\quad 0 &\geq \epsilon \sum_{i=1}^{\frac{N-1}{2}} \sum_{j=\frac{N+1}{2}}^{N} (z_{i1} - z_{j1}) \exp(-r\epsilon(z_{i1} + z_{j1}))
\end{aligned}$$

Recall that $z_{i1} \leq z_{j1}$ as $i \in \{1, ..., \frac{N-1}{2}\}$ while $j \in \{\frac{N+1}{2}, ..., N\}$, i.e. $i < j$, and that $z_{i1}, z_{j1} \in \{-1, -\gamma, \gamma, 1\}$ with $\gamma \in (0, 1)$. Therefore the term $(z_{i1} - z_{j1})$ can take the following values: -2, $-1-\gamma$, $-1+\gamma$, -2γ and 0, i.e. $z_{i1} - z_{j1} \leq 0\ \forall\ i, j$. Furthermore, we can conclude that at least one factor is strictly smaller than zero as long as there exists at least one pair of individuals (i, j) with different benefits if the first-period project is realized $(z_{i1} \neq z_{j1})$, i.e. $\frac{A_W}{A_L}$ is strictly decreasing in this case. We obtain the following proposition:

Proposition 4
Assume a society with $N \geq 2$ individuals who have benefits $z_{i1} \in \{-1, -\gamma, \gamma, 1\}$ from a durable project in $t = 1$ and assume that there exists at least one individual who suffers from the project in $t = 1$ and that there exist two individuals with different first-period preferences.

Then there exists a critical level of risk aversion R, such that MV is better than SM if individuals' utility can be described by $f(x) = -\exp(-rx)$ with parameter $r \geq R$.

Proposition 4 indicates that MV is interim superior to SM if individuals are sufficiently risk-averse and if there is at least one individual who suffers from the first-period project. Note that we have also assumed throughout this section that the project is adopted in $t = 1$. The critical value R depends on N, ϵ and γ. The proof of Proposition 4 is given in Appendix A.

Because of the monotonicity of $\frac{A_W}{A_L}$ with respect to r, Proposition 4 also indicates that MV becomes better the larger r is, i.e. the more risk-averse individuals are. The critical level of risk aversion $R = R(N, \epsilon, \gamma)$ is determined by the requirement that the maximal value of $\frac{A_W}{A_L}$ does not exceed $Q(N)$, i.e.

$$R = \arg\min_{r}\{r : \frac{A_W}{A_L}(r) \leq Q(N) \quad \forall \{z_{i1}\}_{i=1}^{N}\}.$$

Using the approximation of $Q(N)$ we derive an algebraic expression for an upper boundary $\hat{R}$ of R that gives us the following information: MV is better than SM for all $r \geq \hat{R}$ and may be better for some smaller r, depending on the size of the legislature N. The result is given in the following corollary:

Corollary 2

Given the situation of Proposition 4 with $N \equiv 3(4)$, *we obtain as upper boundary* $\hat{R}$ *for the critical value* R

$$\hat{R}(N, \epsilon, \gamma) = \frac{1}{2\epsilon\gamma} \log\Big(\frac{N}{\sqrt{2}} + \frac{3\sqrt{2} - 4}{2(\sqrt{2} - 1)}\Big).$$

The proof of Corollary 2 is given in Appendix A.

Observations:

- If the first-period project depreciates rapidly, i.e. ϵ is very small, then $\hat{R}$ becomes larger. The individuals have to be more risk-averse for MV to be preferable over SM.

- If individuals either benefit or suffer strongly from the first-period project ($\gamma \to 1$) then the loss of belonging to the project losers and therefore to the minority is significant even for slightly risk-averse individuals. Moreover, the lasting utility gain from the first-period project is higher for voting winners. The MV voting scheme becomes more efficient even if r - the rate of absolute risk aversion - is small ($\hat{R}(N, \epsilon, \gamma)$ decreases as γ goes up).

- $\hat{R}(N, \epsilon, \gamma)$ is increasing in N. If the committee becomes larger, then the individuals have to be more risk-averse for MV to be superior to SM. An immediate consequence is that MV should only be applied on middle-sized or small committees in this context which indeed seems plausible.

In Appendix B a number of tables are provided that give values of $\hat{R}(N, \epsilon, \gamma)$ for large N. There are also results of the direct comparison $\max(\frac{A_W}{A_L}) \leq Q(N)$ in the case of small N, since in these cases the approximation via $\sqrt{2} - 1$ is rather poor. The tables indicate that especially for a high ϵ, i.e. when the project is durable, the necessary degree of risk aversion $\hat{R}(N, \epsilon, \gamma)$ is very low (< 8). Considering infrastructure projects, a high ϵ seems to be plausible.

In the next section we set out to derive an ex ante comparison between MV and SM, i.e. when first-period benefits are not yet distributed.

5 The Main Result for the First Period

In this section we derive the ex ante welfare comparison for the utility function with absolute risk aversion, given by $f(x) = -\exp(-r\,x)$, $r > 0$. We already know that MV is superior to SM if $a_1 = 1$, $\epsilon \neq 0$ and individuals are sufficiently risk-averse. On the other hand, we have shown in section 4.2 that MV is worse than SM if $a_1 = 0$ or $\epsilon = 0$, i.e. in the case where there is no utility impact from the first-period project. The remaining question is whether MV could be ex ante socially more efficient than SM, i.e. when $\{z_{i1}\}_{i=1}^N$ are not yet known and all cases could occur. Can the losses incurred by non-durable projects or non-accepted projects accruing between MV and SM be compensated by the gain that the minority experiences?

According to Lemma 1 we can use the ex ante expected welfare of the second period for the comparison. According to Lemma 2 it is sufficient to compare

$$\Delta\mathbb{E}_1[W_2^{MV}] \text{ and } \Delta\mathbb{E}_1[W_2^{SM}].$$

5.1 Ex-Ante Perspective for Constant Absolute Risk Aversion

We have the following situation: $z_{i1} \in \{-1, -\gamma, \gamma, 1\}$ with $\gamma \in (0,1)$, where each value has the same probability, i.e. $\frac{1}{4}$. We assume that the first-period project is durable, i.e. $\epsilon > 0$. The set of all project losers is denoted by I^-:

$$I^- := \{i \in \{1, ..., N\} : z_{i1} < 0\}$$

We distinguish between two cases.

1. $|I^-| \leq \frac{N-1}{2}$.
2. $|I^-| \geq \frac{N+1}{2}$.

In the first case, the proposal in $t = 1$ is adopted, i.e. $a_1 = 1$, in case two the status quo prevails.

In this section we do not assume that individuals are ordered by their benefits. We still apply maximal magnanimity.

5.1.1 The Case $|I^-| \leq \frac{N-1}{2}$

Since utility in the first period is the same under both voting schemes, we only examine the expected utility in the second period. To obtain $\Delta\mathbb{E}_1[W_2^{MV}]$ and $\Delta\mathbb{E}_1[W_2^{SM}]$, we will sum up the expected utility of the second period if the second project is undertaken for all possible realizations of first-period preferences $\{z_{i1}\}_{i=1}^N$. The MV scheme yields

$$
\begin{aligned}
&\Delta\mathbb{E}_1[W_2^{MV}] \\
= \;& \tfrac{1}{4^N}\Big\{ \textstyle\sum_{\beta_1=0}^{\frac{N-1}{2}} \sum_{\beta_2=0}^{\frac{N-1}{2}-\beta_1} \sum_{\alpha=0}^{\frac{N-1}{2}-\beta_1-\beta_2} \binom{N}{\beta_1}\binom{N-\beta_1}{\beta_2}\binom{N-\beta_1-\beta_2}{\alpha} \\
&\cdot \;\Big[\beta_1[P(\tfrac{N-1}{2})F(-\gamma,1) + (1-2P(\tfrac{N-1}{2}))F(-\gamma,0) + (P(\tfrac{N-1}{2})-1)F(-\gamma,-1)] \\
&\quad +\beta_2[P(\tfrac{N-1}{2})F(-1,1) + (1-2P(\tfrac{N-1}{2}))F(-1,0) + (P(\tfrac{N-1}{2})-1)F(-1,-1)] \\
&\quad +\alpha[P(\tfrac{N-1}{2})F(\gamma,1) + (1-2P(\tfrac{N-1}{2}))F(\gamma,0) + (P(\tfrac{N-1}{2})-1)F(\gamma,-1)] \\
&\quad +(\tfrac{N-1}{2}-\beta_1-\beta_2-\alpha)[P(\tfrac{N-1}{2})F(1,1) + (1-2P(\tfrac{N-1}{2}))F(1,0) \\
&\qquad\qquad +(P(\tfrac{N-1}{2})-1)F(1,-1)] \\
&\quad +\tfrac{N+1}{2}\,\tfrac{1}{2}[F(1,1)-F(1,-1)]\Big] \\
&+ \textstyle\sum_{\beta_1=0}^{\frac{N-1}{2}} \sum_{\beta_2=0}^{\frac{N-1}{2}-\beta_1} \sum_{\alpha=\frac{N+1}{2}-\beta_1-\beta_2}^{N-\beta_1-\beta_2} \binom{N}{\beta_1}\binom{N-\beta_1}{\beta_2}\binom{N-\beta_1-\beta_2}{\alpha} \\
&\cdot \;\Big[\beta_1[P(\tfrac{N-1}{2})F(-\gamma,1) + (1-2P(\tfrac{N-1}{2}))F(-\gamma,0) + (P(\tfrac{N-1}{2})-1)F(-\gamma,-1)] \\
&\quad +\beta_2[P(\tfrac{N-1}{2})F(-1,1) + (1-2P(\tfrac{N-1}{2}))F(-1,0) + (P(\tfrac{N-1}{2})-1)F(-1,-1)] \\
&\quad +(\tfrac{N-1}{2}-\beta_1-\beta_2)[P(\tfrac{N-1}{2})F(\gamma,1) + (1-2P(\tfrac{N-1}{2}))F(\gamma,0) \\
&\qquad\qquad +(P(\tfrac{N-1}{2})-1)F(\gamma,-1)] \\
&\quad +(\alpha-\tfrac{N-1}{2}+\beta_1+\beta_2)\tfrac{1}{2}[F(\gamma,1)-F(\gamma,-1)] \\
&\quad +(N-\alpha-\beta_1-\beta_2)\,\tfrac{1}{2}[F(1,1)-F(1,-1)]\Big]\Big\}
\end{aligned}
$$

where α denotes the number of individuals i with $z_{i1}=\gamma$ and β_1 (β_2) denotes the number of individuals i with $z_{i1}=-\gamma$ ($z_{i1}=-1$).

Under the simple majority rule we obtain

$$
\begin{aligned}
&\Delta\mathbb{E}_1[W_2^{SM}] \\
= \;& \frac{1}{4^N}\sum_{\beta_1=0}^{\frac{N-1}{2}} \sum_{\beta_2=0}^{\frac{N-1}{2}-\beta_1} \sum_{\alpha=0}^{N-\beta_1-\beta_2} \binom{N}{\beta_1}\binom{N-\beta_1}{\beta_2}\binom{N-\beta_1-\beta_2}{\alpha} \\
&\cdot \;\Big[\beta_1[P(N)F(-\gamma,1) + (1-2P(N))F(-\gamma,0) + (P(N)-1)F(-\gamma,-1)] \\
&\quad +\beta_2[P(N)F(-1,1) + (1-2P(N))F(-1,0) + (P(N)-1)F(-1,-1)] \\
&\quad +\alpha[P(N)F(\gamma,1) + (1-2P(N))F(\gamma,0) + (P(N)-1)F(\gamma,-1)] \\
&\quad +(N-\beta_1-\beta_2-\alpha)[P(N)F(1,1) + (1-2P(N))F(1,0) + (P(N)-1)F(1,-1)]\Big]
\end{aligned}
$$

5.1.2 The Case $|I^-| \geq \frac{N+1}{2}$

If the majority has negative benefits in connection with the first project, the status quo prevails, i.e. $a_1 = 0$. In this case $F(z_{i1}, z_{i2})$ reduces to $F(z_{i2})$. If we set $z_{i1} = 0$ for all individuals i, we obtain the same result as in the case $a_1 = 0$. Therefore we identify $F(z_{i2})$ with $F(0, z_{i2})$ and obtain

$$\begin{aligned}
\Delta \mathbb{E}_1[W_2^{MV}] &= \frac{1}{2}\Big\{ \frac{N-1}{2}\Big[P(\frac{N-1}{2})F(0,1) + \left(1 - 2P(\frac{N-1}{2})\right)F(0,0) \\
&\quad + \left(P(\frac{N-1}{2}) - 1\right)F(0,-1)\Big] + \frac{N+1}{2}\frac{1}{2}[F(0,1) - F(0,-1)]\Big\} \\
\Delta \mathbb{E}_1[W_2^{SM}] &= \frac{1}{2}N\Big[P(N)F(0,1) + (1 - 2P(N))F(0,0) + (P(N) - 1)F(0,-1)\Big]
\end{aligned}$$

The factor $\frac{1}{2}$ represents the probability that at least $\frac{N+1}{2}$ individuals have negative first-period benefits.

5.2 Comparison

5.2.1 Existence of a critical value of risk aversion R^*

With the results of section 5.1.1 and section 5.1.2 we can show that there is indeed a critical value of risk aversion R^* such that MV is ex ante superior to SM if the individuals' level of risk aversion r is greater than R^*. If individuals fear the potential loss from belonging to the minority under SM, then MV becomes more attractive to them. We first show the existence of R^*.

Proposition 5

Suppose a society of N members, whose utility in both periods ($t = 1, 2$) can be described by $f(x) = -\exp(-r\,x)$ with $r > 0$. Then there exists a critical value R^* such that MV is ex ante superior to SM if $r \geq R^*$.

The proof of Proposition 5 is given in Appendix A. The intuition of this result is clear: the more risk-averse individuals are, the more important is the potential loss under SM if they belong to the minority and hence, the more attractive is MV.

5.2.2 Calculation of R^*

Proposition 5 suggests that the welfare gain due to MV, i.e. $\Delta\mathbb{E}_1[W_2^{MV}] - \Delta\mathbb{E}_1[W_2^{SM}]$, is monotonically increasing in risk aversion. We did not need this property to show the existence of a critical level of risk aversion at which MV becomes more attractive than SM. We now turn to the numerical determination of R^* for which monotonicity of the welfare gain is needed. A constructive approach for the critical level of risk aversion R^* is given in the following proposition.

Proposition 6

Let R^* be determined by the following equation

$$\Delta\mathbb{E}_1[W_2^{MV}](R^*) - \Delta\mathbb{E}_1[W_2^{SM}](R^*) = 0.$$

Then the MV scheme becomes socially more efficient than SM if risk aversion r of all individuals is greater than R^*.

The proof of Proposition 6 follows directly from Lemma 4 and Lemma 5 which are described below. To formulate the lemmas, we start with equation (6) from the proof of Proposition 5 that describes the expected welfare differences and which takes the following form:

$$\begin{aligned} &\Delta\mathbb{E}_1[W_2^{MV}] - \Delta\mathbb{E}_1[W_2^{SM}] \\ &= \frac{C}{r}(e^{-r} + e^{r} - 2)\Big[c_1 e^{-r\epsilon} + c_2 e^{-r\epsilon\gamma} + c_3 e^{r\epsilon} + c_4 e^{r\epsilon\gamma} + c_5\Big] \end{aligned}$$

with some constants C, c_1, c_2, c_3, c_4 and c_5. Regarding these constants the following lemma holds.

Lemma 4

The constants C, c_3, c_4 are positive, c_1, c_5 are negative. Furthermore, $c_1 < c_2 < c_3 = c_4$.

A proof is given in Appendix A.

Lemma 4 implies that the utility gain for voting losers under minority voting with benefits $z_{i1} = -\gamma$ or $z_{i1} = -1$ has a positive effect on the welfare difference ($c_3, c_4 > 0$), while the utility loss of minority winners under MV (with benefits $z_{i1} = 1$) has a negative effect ($c_1 < 0$). Moreover, the utility gain through strategic voting does not compensate these project winners for the utility loss of being a voting winner.

Individuals with $z_{i1} = \gamma$ are more likely to switch sides due to strategic voting than individuals with $z_{i1} = 1$. The resulting utility gain under MV is higher. Hence, their overall utility change might even have a positive effect on the difference $\Delta\mathbb{E}_1[W_2^{MV}] - \mathbb{E}_1[\Delta W^{SM}]_2$. However, numerical results suggest that individuals with $z_{i1} = \gamma$ also have a negative utility change from SM to MV, i.e. $c_2 < 0$.

With this result we can show the monotonicity of $\Delta\mathbb{E}_1[W_2^{MV}] - \mathbb{E}_1[\Delta W_2^{SM}]$.

Lemma 5
The welfare gain due to MV, i.e. $\Delta\mathbb{E}_1[W_2^{MV}] - \Delta\mathbb{E}_1[W_2^{SM}]$, is monotonically increasing in r.

The proof of Lemma 5 is given in Appendix A.

Lemma 5 finally proves Proposition 6: If we can determine R^* such that $\Delta\mathbb{E}_1[W_2^{MV}](R^*) - \Delta\mathbb{E}_1[W_2^{SM}](R^*) = 0$, we know that MV is better than SM if the absolute risk aversion of all individuals is greater than R^*. We give some examples in the next section.

5.2.3 Examples

In this subsection we provide some numerical examples and focus on $N = 3$, i.e. the society consists of 3 individuals. Rearranging equation (6) yields

$$\Delta\mathbb{E}_1[W_2^{MV}] - \Delta\mathbb{E}_1[W_2^{SM}] = \frac{1}{64\,r}(e^{-r} + e^{r} - 2)\Big[-8.5e^{-r\epsilon} - 5.5e^{-r\epsilon\gamma} + 3e^{r\epsilon} + 3e^{r\epsilon\gamma} - 8\Big].$$

We assume that $r > 0$ and obtain

$$\begin{aligned} \Delta\mathbb{E}_1[W_2^{MV}] - \Delta\mathbb{E}_1[W_2^{SM}] &= 0 \\ \Leftrightarrow -8.5e^{-r\epsilon} - 5.5e^{-r\epsilon\gamma} + 3e^{r\epsilon} + 3e^{r\epsilon\gamma} - 8 &= 0 \end{aligned}$$

We give sample plots for R^* (see figure 2).

The upper line indicates the case $\epsilon = 0.1$, the middle line represents $\epsilon = 0.5$, and the lower line is $\epsilon = 0.9$. Note that we can make the same observations as in section 4.4 concerning $R(N, \epsilon, \gamma)$: R^* is decreasing in γ and ϵ.

Graphics for $N = 5$ and $N = 7$ are given in Appendix C.

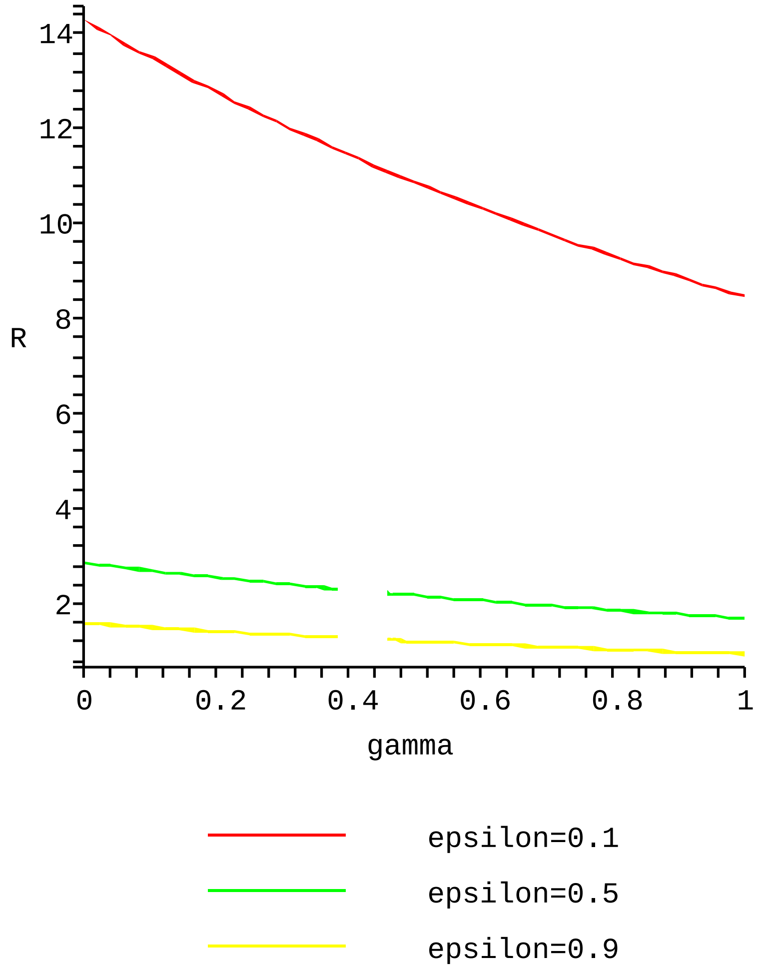

Figure 2: R^* for $N = 3$, $\epsilon \in \{0.1, 0.5, 0.9\}$, $\gamma \in [0, 1]$

6 MV and Important Properties of SM

It is useful to recapitulate the axiomatic properties of MV in relation to SM. The (repeated) simple majority rule has some important properties: it satisfies neutrality, i.e. all alternatives are treated equally, and anonymity in the sense that every vote has the same impact on the outcome of the election, i.e. all voters are treated equally. Furthermore, the decision function based on the simple majority rule is well defined, single valued and satisfies positive responsiveness, i.e. if a group is indifferent or in favor of an alternative x and all preferences remain the same except the preferences of one single individual that change in favor of alternative x then the group decision also becomes favorable to x. In fact, the simple majority rule is the only voting scheme fulfilling all these properties (see May (1952)). Moreover, it is irrelevant whether the voting process is open or closed (in our setting). Individuals always vote in favor of their preferences as they do not have to gain anything by deviating from this strategy. Furthermore, knowing the voting behaviour of other committee members does not change the own voting or the output. We will analyze MV regarding these properties.

Neutrality: The MV scheme does satisfy neutrality as the status quo and the projects proposed are treated equally in each period. This property is always satisfied when two alternatives are competing and 50% are needed for acceptance.

Anonymity: MV satisfies anonymity in the first period and anonymity within the group that keeps the voting right in the second period. Moreover, as all individuals are ex ante identical, they have the same probability of keeping or loosing their voting right and hence, MV satisfies ex ante anonymity.

Positive responsiveness and well defined, single valued decision function: In the first period the MV scheme satisfies these properties as the first-period project is realized if and only if the number of individuals with non-zero benefits from this project is greater or equal than $\frac{N+1}{2}$. In the second period we have to make the same distinction as for anonymity: MV satisfies positive responsiveness and well defined, single valued decision function for the group of individuals who have voting right in $t = 2$, but not for the individuals who loose their second-period voting right.

Secret ballots: In our setting voting takes place openly. The minority has to be identified in order to determine who keeps and who looses the voting right in the second period. Since secret ballots may be a desirable axiom, we give an alternative formulation of MV with secret ballots.

Secret ballot under MV:

- There are two projects that will be realized in two consecutive periods $t = 1, 2$.
- The voting takes place in period $t = 0$. The committee decides about both projects simultaniously. The votes of one individual have to be linked, e.g. making the crosses on one sheet of paper.
- Evaluation:
    1. The votes for the first project are counted and the project decision is determined. If there is a tally, the tie-breaking rule is used, i.e. every alternative wins with probability $\frac{1}{2}$.
    2. The ballots of the minority are identified. Their second vote is identified using their ballot papers.
    3. The decision over the second project is determined.

Under this voting scheme we assume that an individual's decisions on both projects are handed in together. Therefore the minority's second-project choice can be identified without knowing which person belongs to the voting losers. It is sufficient to know the first-project vote and the linked second-project vote. The most important requirement is that the second decision is already identified when the first decision takes place. This seems plausible as our introductory examples illustrate (e.g. the infrastructure example): the projects are pre-planned and linked in certain ways which translates into the durable impact of the first-period project in our setting.

To sum up MV fulfills two of the five main properties of SM, i.e. neutrality and secret ballots across both periods. In the first period all properties are satisfied. Because of the restriction of the second-period voting right to voting losers, the MV scheme violates anonymity, positive responsiveness and well defined, single valued decision function in the second period for the whole committee, but not for the group of individuals who keep the voting right. Finally, anonymity is fulfilled from an ex ante point of view.

7 Conclusion

We have introduced a new 2-period voting scheme that strengthens the impact of first-period voting losers. Moreover, this scheme improves the ex ante utility of the whole society if individuals are sufficiently risk-averse. The minority voting scheme, and especially the support of minorities, deserves further research. For instance, it is useful to examine the case where the benefits $\{z_{i1}\}_{i=1}^{N}$ are not common knowledge. Intuitively, individuals will not vote strategically in that case as they do not know which alternative will obtain a majority of votes. This behavior leads to minorities of a significantly smaller size than the majority which tends to weaken MV relative to SM. However, in the case where all individuals have non-negative benefits MV becomes more favorable than in the current set-up as all individuals would keep their voting rights (because of unanimity). Overall, the question how extensions of our model shift the balance between SM and MV is an entire research programme.

8 Appendix A

Proof of Proposition 1

The first point is obvious, as abstention eliminates the voting right without any benefit.

Suppose the number of project losers $k \geq \frac{N+1}{2}$ and there are more than $\frac{N+1}{2}$ project losers who vote for $a_1 = 0$ in the equilibrium, i.e. $N^w - N^l > 1$. If individual i belongs to the majority: $i \in N^w$, then assuming that all other individuals will play their equilibrium strategies deviating from his strategy by voting for the new project increases individual i's payoff: the status quo still prevails and therefore $u_{i1} = 0$ but i keeps his voting right in $t = 2$. Given $|N^w - N^l| = 1$, no individual in the majority has an incentive to deviate, as such a switch will change the committee decision, and the individual under consideration still loses his voting right.

The same arguments hold *mutatis mutandis* for the third point of Proposition 1.

$\blacksquare$

Proof of Proposition 2

We first rewrite $\Delta\mathbb{E}_2[W_2^{MV}]$ using the definition $F(z_{i1}, z_{i2}) = \int f(\epsilon a_1 z_{i1} + z_{i2})dz_{i2}$.

$$
\begin{aligned}
& \Delta\mathbb{E}_2[W_2^{MV}] \\
= \; & \textstyle\sum_{i=1}^{\frac{N-1}{2}} \Big[P(\frac{N-1}{2}) \int_0^1 f(\epsilon z_{i1} + z_{i2})dz_{i2} + (1 - P(\frac{N-1}{2})) \int_{-1}^0 f(\epsilon z_{i1} + z_{i2})dz_{i2}\Big] \\
& + \tfrac{1}{2} \textstyle\sum_{i=\frac{N+1}{2}}^{N} \int_{-1}^{1} f(\epsilon z_{i1} + z_{i2})dz_{i2} \\
= \; & \textstyle\sum_{i=1}^{\frac{N-1}{2}} \Big[P(\frac{N-1}{2})(F(z_{i1}, 1) - F(z_{i1}, 0)) + (1 - P(\frac{N-1}{2}))(F(z_{i1}, 0) - F(z_{i1}, -1))\Big] \\
& + \tfrac{1}{2} \textstyle\sum_{i=\frac{N+1}{2}}^{N} (F(z_{i1}, 1) - F(z_{i1}, -1)) \\
= \; & \textstyle\sum_{i=1}^{\frac{N-1}{2}} \Big[P(\frac{N-1}{2})F(z_{i1}, 1) - (2P(\frac{N-1}{2}) - 1)F(z_{i1}, 0) + (P(\frac{N-1}{2}) - 1)F(z_{i1}, -1)\Big] \\
& + \tfrac{1}{2} \textstyle\sum_{i=\frac{N+1}{2}}^{N} (F(z_{i1}, 1) - F(z_{i1}, -1))
\end{aligned}
$$

Similarly we obtain

$$
\Delta\mathbb{E}_2[W_2^{SM}] = \sum_{i=1}^{N} \Big[P(N)F(z_{i1}, 1) - (2P(N) - 1)F(z_{i1}, 0) + (P(N) - 1)F(z_{i1}, -1)\Big]
$$

A comparison yields

$$
\begin{aligned}
& \Delta\mathbb{E}_2[W_2^{MV}] \geq \Delta\mathbb{E}_2[W_2^{SM}] \\
\Leftrightarrow\ & \textstyle\sum_{i=1}^{\frac{N-1}{2}} \Big[F(z_{i1},1)[P(\frac{N-1}{2}) - P(N)] - F(z_{i1},0)[2P(\frac{N-1}{2}) - 2P(N)] \\
& +F(z_{i1},-1)[P(\frac{N-1}{2}) - P(N)]\Big] + \textstyle\sum_{i=\frac{N+1}{2}}^{N} \Big[F(z_{i1},1)[\frac{1}{2} - P(N)] \\
& -F(z_{i1},0)[1-2P(N)] + F(z_{i1},-1)[\frac{1}{2} - P(N)]\Big] \qquad \geq 0
\end{aligned}
$$

Rearranging terms proves Proposition 2.

■

Proof of Lemma 3

The function f is increasing and concave (in $\epsilon a_1 z_{i1} + z_{i2}$ as a whole argument and both in z_{i1} and z_{i2}, i.e. $\frac{\partial f(x)}{\partial x} \geq 0$ and $\frac{\partial^2 f(x)}{\partial x^2} \leq 0$. Therefore we obtain $f(x+1) - 2f(x) + f(x-1) \leq 0$ because

$$f(x+1) - f(x) \leq f(x) - f(x-1).$$

The integral $F(z_{i1}, z_{i2})$ is a convex function in z_{i2} because $\frac{\partial^2 F(z_{i1},z_{i2})}{\partial z_{i2}^2} = \frac{\partial f(\epsilon a_1 z_{i1} + z_{i2})}{\partial z_{i2}} \geq 0$. Hence,

$$
\begin{aligned}
F(z_{i1},1) - F(z_{i1},0) &\geq F(z_{i1},0) - F(z_{i1},-1) \\
\Leftrightarrow\ F(z_{i1},1) - 2F(z_{i1},0) + F(z_{i1},-1) &\geq 0 \\
\Leftrightarrow\ A(z_{i1}) &\geq 0
\end{aligned}
$$

Furthermore, we can conclude that $A(z_{i1}) \geq A(z_{j1}) \Leftrightarrow z_{i1} \leq z_{j1}$ because
$f(\epsilon z_{i1} + 1) - 2f(\epsilon z_{i1}) + f(\epsilon z_{i1} - 1) \leq 0$ since f is concave.

■

Proof of Proposition 3

We have to show that

$$\frac{N-1}{2}[P(\frac{N-1}{2})-P(N)]-\frac{N+1}{2}[P(N)-\frac{1}{2}]<0$$

Because of the definition of $P(w)$ this inequality is equivalent to

$$P_{cond}(N):=\frac{N-1}{2^{\frac{N+1}{2}}}\binom{\frac{N-3}{2}}{\lfloor\frac{N-3}{4}\rfloor}-\frac{N}{2^N}\binom{N-1}{\frac{N-1}{2}}<0 \tag{5}$$

We show this inequality via complete induction. Note that $P_{cond}(N)$ is oscillating. That means $P_{cond}(N)$ is decreasing with respect to N with $N\equiv 3(4)$ and decreasing with respect to N with $N\equiv 1(4)$, but if we observe two odd numbers in a row then the function may either decrease or increase. We therefore make a double induction.

1. The inequality holds for $N=3$, $N=5$ and $N=7$:

- $\frac{2}{2^2}\binom{0}{0}-\frac{3}{2^3}\binom{2}{1}=-\frac{1}{4}<0$
- $\frac{4}{2^3}\binom{1}{0}-\frac{5}{2^5}\binom{4}{2}=-\frac{7}{16}<0$
- $\frac{6}{2^4}\binom{2}{1}-\frac{7}{2^7}\binom{6}{3}=-\frac{11}{32}<0$

2. Suppose inequality (5) holds for $N\geq 3$. If $P_{cond}(N+4)\leq P_{cond}(N)$ then inequality (5) also holds for $N+4$. We show this step for $N\equiv 3(4)$ and $N\equiv 1(4)$. This proves the proposition.

$$P_{cond}(N+4)=\frac{N+3}{2^{\frac{N+5}{2}}}\binom{\frac{N+1}{2}}{\lfloor\frac{N+1}{4}\rfloor}-\frac{N+4}{2^{N+4}}\binom{N+3}{\frac{N+3}{2}}$$

We obtain $P_{cond}(N+4)\leq P_{cond}(N)$ if and only if

$$\begin{aligned}
&\tfrac{N+3}{2^{\frac{N+5}{2}}}\tbinom{\frac{N+1}{2}}{\lfloor\frac{N+1}{4}\rfloor}-\tfrac{N+4}{2^{N+4}}\tbinom{N+3}{\frac{N+3}{2}} &\leq\ & \tfrac{N-1}{2^{\frac{N+1}{2}}}\tbinom{\frac{N-3}{2}}{\lfloor\frac{N-3}{4}\rfloor}-\tfrac{N}{2^N}\tbinom{N-1}{\frac{N-1}{2}}\\
\Leftrightarrow\ & (N+3)2^{\frac{N+3}{2}}\tbinom{\frac{N+1}{2}}{\lfloor\frac{N+1}{4}\rfloor}-(N+4)\tbinom{N+3}{\frac{N+3}{2}} &\leq\ & (N-1)2^{\frac{N+7}{2}}\tbinom{\frac{N-3}{2}}{\lfloor\frac{N-3}{4}\rfloor}-16N\tbinom{N-1}{\frac{N-1}{2}}\\
\Leftrightarrow\ & 16N\tbinom{N-1}{\frac{N-1}{2}}-(N+4)\tbinom{N+3}{\frac{N+3}{2}} &\leq\ & (N-1)2^{\frac{N+7}{2}}\tbinom{\frac{N-3}{2}}{\lfloor\frac{N-3}{4}\rfloor}-(N+3)2^{\frac{N+3}{2}}\tbinom{\frac{N+1}{2}}{\lfloor\frac{N+1}{4}\rfloor}
\end{aligned}$$

First we analyse the LHS.

$$\begin{aligned}
& 16N\binom{N-1}{\frac{N-1}{2}} - (N+4)\binom{N+3}{\frac{N+3}{2}} \\
= \; & 16N\binom{N-1}{\frac{N-1}{2}} - (N+4)\frac{(N+3)!}{(\frac{N+3}{2})!^2} \\
= \; & 16N\binom{N-1}{\frac{N-1}{2}} - (N+4)\frac{(N+3)(N+2)(N+1)N\cdot(N-1)!}{(\frac{N+3}{2})^2(\frac{N+1}{2})^2(\frac{N-1}{2})!^2} \\
= \; & \binom{N-1}{\frac{N-1}{2}}\Big[16N - (N+4)\frac{(N+3)(N+2)(N+1)N}{(\frac{N+3}{2})^2(\frac{N+1}{2})^2}\Big] \\
= \; & \binom{N-1}{\frac{N-1}{2}}\Big[16N - \frac{16N(N+4)(N+2)}{(N+3)(N+1)}\Big] \\
= \; & 16N\binom{N-1}{\frac{N-1}{2}}\Big(1 - \frac{(N+4)(N+2)}{(N+3)(N+1)}\Big) \\
< \; & 0
\end{aligned}$$

The LHS is smaller than zero for all odd numbers N. We show that the RHS is greater or equal to zero for all N with N odd. For this purpose we first transform the RHS.

$$\begin{aligned}
& 2^{\frac{N+3}{2}}\Big[4(N-1)\binom{\frac{N-3}{2}}{\lfloor\frac{N-3}{4}\rfloor} - (N+3)\binom{\frac{N+1}{2}}{\lfloor\frac{N+1}{4}\rfloor}\Big] \\
= \; & 2^{\frac{N+3}{2}}\Big[4(N-1)\binom{\frac{N-3}{2}}{\lfloor\frac{N-3}{4}\rfloor} - (N+3)\frac{(\frac{N+1}{2})!}{\lfloor\frac{N+1}{4}\rfloor!(\frac{N+1}{2}-\lfloor\frac{N+1}{4}\rfloor)!}\Big] \\
= \; & 2^{\frac{N+3}{2}}\Big[4(N-1)\binom{\frac{N-3}{2}}{\lfloor\frac{N-3}{4}\rfloor} - (N+3)\frac{(\frac{N+1}{2})(\frac{N-1}{2})\cdot(\frac{N-3}{2})!}{\lfloor\frac{N+1}{4}\rfloor\cdot\lfloor\frac{N-3}{4}\rfloor!\cdot(2+\frac{N-3}{2}-(\lfloor\frac{N-3}{4}\rfloor+1))!}\Big] \\
= \; & 2^{\frac{N+3}{2}}\Big[4(N-1)\binom{\frac{N-3}{2}}{\lfloor\frac{N-3}{4}\rfloor} - (N+3)\frac{(\frac{N+1}{2})(\frac{N-1}{2})\cdot(\frac{N-3}{2})!}{\lfloor\frac{N+1}{4}\rfloor\cdot\lfloor\frac{N-3}{4}\rfloor!\cdot(\frac{N-1}{2}-\lfloor\frac{N-3}{4}\rfloor)\cdot(\frac{N-3}{2}-\lfloor\frac{N-3}{4}\rfloor)!}\Big] \\
= \; & 2^{\frac{N+3}{2}}\binom{\frac{N-3}{2}}{\lfloor\frac{N-3}{4}\rfloor}\Big[4(N-1) - (N+3)\frac{(\frac{N+1}{2})(\frac{N-1}{2})}{\lfloor\frac{N+1}{4}\rfloor(\frac{N+1}{2}-\lfloor\frac{N-3}{4}\rfloor)}\Big]
\end{aligned}$$

For the last step we need to distinguish between $N \equiv 1(4)$ and $N \equiv 3(4)$.

In the first case we have $\lfloor\frac{N+1}{4}\rfloor = \frac{N-1}{4}$, $\lfloor\frac{N-3}{4}\rfloor = \frac{N-5}{4}$ and $\frac{N+1}{2} - \lfloor\frac{N-3}{4}\rfloor = \frac{N+7}{4}$.

If $N \equiv 3(4)$, i.e. $\frac{N+1}{4} \notin \mathbb{N}$, we obtain $\lfloor\frac{N+1}{4}\rfloor = \frac{N+1}{4}$, $\lfloor\frac{N-3}{4}\rfloor = \frac{N-3}{4}$ and $\frac{N+1}{2} - \lfloor\frac{N-3}{4}\rfloor = \frac{N+5}{4}$.

- Calculation of the RHS with $N \equiv 1(4)$.

$$\begin{aligned} RHS &= 2^{\frac{N+3}{2}} \binom{\frac{N-3}{2}}{\lfloor \frac{N-3}{4} \rfloor} \Big[4(N-1) - (N+3) \frac{(\frac{N+1}{2})(\frac{N-1}{2})}{(\frac{N-1}{4})(\frac{N+7}{4})} \Big] \\ &= 2^{\frac{N+3}{2}} \binom{\frac{N-3}{2}}{\lfloor \frac{N-3}{4} \rfloor} \Big[4(N-1) - \frac{4(N+1)(N+3)}{(N+7)} \Big] \\ &\geq 0 \end{aligned}$$

- Calculation of the RHS with $N \equiv 3(4)$.

$$\begin{aligned} RHS &= 2^{\frac{N+3}{2}} \binom{\frac{N-3}{2}}{\frac{N-3}{4}} \Big[4(N-1) - (N+3) \frac{(\frac{N+1}{2})(\frac{N-1}{2})}{(\frac{N+1}{4})(\frac{N+5}{4})} \Big] \\ &= 2^{\frac{N+3}{2}} \binom{\frac{N-3}{2}}{\frac{N-3}{4}} \Big[4(N-1) - \frac{4(N-1)(N+3)}{(N+5)} \Big] \\ &= 2^{\frac{N+3}{2}} \binom{\frac{N-3}{2}}{\frac{N-3}{4}} 4(N-1) \Big[1 - \frac{N+3}{N+5} \Big] \\ &\geq 0 \end{aligned}$$

if $N \geq 5$. Since we have shown that $P_{cond}(N) < 0$ for $N = 3, 5, 7$ this inequality completes the proof.

■

Approximation with Normal Distribution

Suppose X is a binomially distributed random variable with parameters n and p. If n becomes sufficiently large and p remains fixed, then X is approximately normally distributed with mean value $\mu_n = np$ and variance $\sigma_n^2 = np(1-p)$, e.g.

$$\mathbb{P}_B[X \leq c] \sim \mathbb{P}_N[Y \leq c + \frac{1}{2}]$$

where $X \sim B(n,p)$, $Y \sim N(\mu_n, \sigma_n^2)$, and $\mathbb{P}[X \leq c]$ denotes the probability that a random variable X adopts a value less or equal to c. The additional term $+\frac{1}{2}$ is called continuity correction.

Now we want to apply this result to $P(N)$. Note that $P(N) = \mathbb{P}[X \geq \frac{N-1}{2}]$ with $X \sim B(N-1, \frac{1}{2})$. We obtain

$$
\begin{aligned}
P(N) &= 1-\mathbb{P}_B[X<\frac{N-1}{2}] \\
&\sim 1-\mathbb{P}_N[Y\leq\frac{N}{2}] \\
&= 1-\frac{1}{\sqrt{\pi\frac{N-1}{2}}}\int_{-\infty}^{\frac{N}{2}}\exp(-\frac{(x-\frac{N-1}{2})^2}{\frac{N-1}{2}})dx
\end{aligned}
$$

Using the substitution rule for integrals we obtain

$$
\frac{1}{\sqrt{\pi\frac{N-1}{2}}}\int_{-\infty}^{\frac{N}{2}}\exp(-\frac{(x-\frac{N-1}{2})^2}{\frac{N-1}{2}})dx=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\frac{1}{2}(\frac{N-1}{2})^{-\frac{1}{2}}}\exp(-y^2)dy
$$

By the same arguments we can express $P(\frac{N-1}{2})$ by

$$
\begin{aligned}
P(\frac{N-1}{2}) &= \mathbb{P}_B[X\geq\lfloor\frac{N-3}{4}\rfloor] \\
&\sim 1-\frac{1}{\sqrt{\pi\lfloor\frac{N-3}{4}\rfloor}}\int_{-\infty}^{\lfloor\frac{N-3}{4}\rfloor+\frac{1}{2}}\exp(-\frac{(x-\lfloor\frac{N-3}{4}\rfloor)^2}{\lfloor\frac{N-3}{4}\rfloor})dx \\
&= 1-\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\frac{1}{2}\lfloor\frac{N-3}{4}\rfloor^{-\frac{1}{2}}}\exp(-y^2)dy
\end{aligned}
$$

The next step is to show that the approximation of the LHS of equation (4) in section 4.3 tends to $\sqrt{2}-1$ as N goes to infinity. Note that the approximation only yields good results if N is sufficiently large. Another problem is that both the LHS of equation (4) and its approximation are oscillating functions.

The Limit of the Approximation

$$
\begin{aligned}
& \frac{(N-1)(P(\frac{N-1}{2})-P(N))}{(N+1)(P(N)-\frac{1}{2})} \\
\simeq\ & \frac{N-1}{N+1}\,\frac{1-\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\frac{1}{2}\lfloor\frac{N-3}{4}\rfloor^{-\frac{1}{2}}}\exp(-y^2)dy-(1-\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\frac{1}{2}(\frac{N-1}{2})^{-\frac{1}{2}}}\exp(-y^2)dy)}{1-\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\frac{1}{2}(\frac{N-1}{2})^{-\frac{1}{2}}}\exp(-y^2)dy-\frac{1}{2}} \\
=\ & \frac{N-1}{N+1}\,\frac{\frac{1}{\sqrt{\pi}}\int_{0}^{\frac{1}{2}(\frac{N-1}{2})^{-\frac{1}{2}}}\exp(-y^2)dy-\frac{1}{\sqrt{\pi}}\int_{0}^{\frac{1}{2}\lfloor\frac{N-3}{4}\rfloor^{-\frac{1}{2}}}\exp(-y^2)dy}{\frac{1}{2}-(\frac{1}{2}+\frac{1}{\sqrt{\pi}}\int_{0}^{\frac{1}{2}(\frac{N-1}{2})^{-\frac{1}{2}}}\exp(-y^2)dy)} \\
=\ & \frac{(N-1)\int_{0}^{\frac{1}{2}\lfloor\frac{N-3}{4}\rfloor^{-\frac{1}{2}}}\exp(-y^2)dy}{(N+1)\int_{0}^{\frac{1}{2}(\frac{N-1}{2})^{-\frac{1}{2}}}\exp(-y^2)dy}-\frac{N-1}{N+1}
\end{aligned}
$$

Note that

- $\int_0^x \exp(-y^2)dy = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)k!} = x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} \pm ...$
- $\lim(a_n\, b_n) = \lim(a_n)\lim(b_n)$ and $\lim(\frac{a_n}{b_n}) = \frac{\lim(a_n)}{\lim(b_n)}$ if $\lim(b_n) \neq 0$.
- $\lim_{N\to\infty}\frac{N-1}{N+1} = \lim_{N\to\infty}\left(\frac{N-3}{N-1}\right)^{\frac{1}{2}} = \lim_{N\to\infty}\left(\frac{N-5}{N-1}\right)^{\frac{1}{2}} = 1$

We assume that $N \equiv 3(4)$ and obtain

$$
\begin{aligned}
& \lim_{N\to\infty}(\frac{(N-1)(P(\frac{N-1}{2})-P(N))}{(N+1)(P(N)-\frac{1}{2})}) \\
=\ & \lim_{N\to\infty}\Big((\frac{N-1}{N+1})\frac{\frac{1}{2}\lfloor\frac{N-3}{4}\rfloor^{-\frac{1}{2}}-\frac{1}{3}(\frac{1}{8})\lfloor\frac{N-3}{4}\rfloor^{-\frac{3}{2}}+\frac{1}{5\cdot 2!}(\frac{1}{32})\lfloor\frac{N-3}{4}\rfloor^{-\frac{5}{2}}-...}{\frac{1}{2}(\frac{N-1}{2})^{-\frac{1}{2}}-\frac{1}{3}(\frac{1}{8})(\frac{N-1}{2})^{-\frac{3}{2}}+\frac{1}{5\cdot 2!}(\frac{1}{32})(\frac{N-1}{2})^{-\frac{5}{2}}-...}-\frac{N-1}{N+1}\Big) \\
=\ & \lim_{N\to\infty}(\frac{N-1}{N+1})\lim_{N\to\infty}(\frac{\frac{1}{2}\lfloor\frac{N-3}{4}\rfloor^{-\frac{1}{2}}-\frac{1}{24}\lfloor\frac{N-3}{4}\rfloor^{-\frac{3}{2}}+\frac{1}{320}\lfloor\frac{N-3}{4}\rfloor^{-\frac{5}{2}}-...}{\frac{1}{2}(\frac{N-1}{2})^{-\frac{1}{2}}-\frac{1}{24}(\frac{N-1}{2})^{-\frac{3}{2}}+\frac{1}{320}(\frac{N-1}{2})^{-\frac{5}{2}}-...})-\lim_{N\to\infty}\frac{N-1}{N+1} \\
=\ & \lim_{N\to\infty}\frac{(N-3)^{\frac{1}{2}}}{(N-1)^{\frac{1}{2}}}\lim_{N\to\infty}(\frac{\frac{1}{2}(\frac{N-3}{4})^{-\frac{1}{2}}-\frac{1}{24}(\frac{N-3}{4})^{-\frac{3}{2}}+\frac{1}{320}(\frac{N-3}{4})^{-\frac{5}{2}}-...}{\frac{1}{2}(\frac{N-1}{2})^{-\frac{1}{2}}-\frac{1}{24}(\frac{N-1}{2})^{-\frac{3}{2}}+\frac{1}{320}(\frac{N-1}{2})^{-\frac{5}{2}}-...})-1 \\
=\ & \lim_{N\to\infty}(\frac{\frac{1}{2}\sqrt{4}-\frac{1}{24}\frac{(N-3)^{-1}}{4^{-\frac{3}{2}}}+\frac{1}{320}\frac{(N-3)^{-2}}{4^{-\frac{5}{2}}}-...}{\frac{1}{2}\sqrt{2}-\frac{1}{24}\frac{(N-1)^{-1}}{2^{-\frac{3}{2}}}+\frac{1}{320}\frac{(N-1)^{-2}}{2^{-\frac{5}{2}}}-...})-1 \\
=\ & \frac{\sqrt{4}}{\sqrt{2}}-1 \\
=\ & \sqrt{2}-1
\end{aligned}
$$

In the case where $N \equiv 1(4)$, e.g. $\lfloor \frac{N-3}{4} \rfloor = \frac{N-5}{4}$, one needs to adjust the intelligent 1 we used in the calculation, i.e. one needs to multiply with $\lim_{N\to\infty}\left(\frac{N-5}{N-1}\right)^{\frac{1}{2}} = 1$ to obtain the same result.

Proof of Proposition 4

Let α_1 and α_2 denote the number of voting winners j with $z_{j1} = 1$ and $z_{j1} = \gamma$ respectively. Let β_1 $(\beta_2, \beta_3, \beta_4)$ denote the number of voting losers i with $z_{i1} = 1$ $(z_{i1} = \gamma, z_{i1} = -\gamma, z_{i1} = -1)$. Note that because of the assumptions either $\beta_4 \geq 1$ or $\beta_3 \geq 1$.

$$\frac{A_W}{A_L} = \frac{(N-1)\left[\alpha_1 e^{-r\epsilon} + \alpha_2 e^{-r\epsilon\gamma}\right]}{(N+1)\left[\beta_1 e^{-r\epsilon} + \beta_2 e^{-r\epsilon\gamma} + \beta_3 e^{r\epsilon\gamma} + \beta_4 e^{r\epsilon}\right]}$$

Since $\exp(x)$ goes to infinity as x goes to infinity and $\exp(-x)$ goes to zero as x goes to infinity, it follows that

$$\lim_{r\to\infty} \frac{A_W}{A_L} = 0$$

Under the assumptions of this proposition $\frac{A_W}{A_L}$ is strictly monotonically decreasing. For every tuple $(z_{i1})_{i=1}^N$, there exists r^* such that MV is better than SM if utility is described by $\exp(-r\epsilon z_{i1} + z_{i2})$ with $r \geq r^*$. Since there are only finite possibilities how preferences can be distributed among the N individuals, there exists an $R = R(N, \epsilon, \gamma)$ such that $\frac{A_W}{A_L}(r) \leq \frac{(N-1)[P(\frac{N-1}{2}) - P(N)]}{(N+1)[P(N) - \frac{1}{2}]} = Q(N)$ for all $r \geq R$ and all possible tuples $(z_{i1})_{i=1}^N$.

■

Proof of Corollary 2

$\hat{R}$ is determined by $\max(\frac{A_W}{A_L}) \leq \sqrt{2} - 1$. Therefore we first determine the combination of benefits that maximizes

$$\frac{A_W}{A_L} = \frac{(N-1)\left[\alpha_1 e^{-r\epsilon} + \alpha_2 e^{-r\epsilon\gamma}\right]}{(N+1)\left[\beta_1 e^{-r\epsilon} + \beta_2 e^{-r\epsilon\gamma} + \beta_3 e^{r\epsilon\gamma} + \beta_4 e^{r\epsilon}\right]}.$$

First, β_4 has to be zero because the denominator has to be as small as possible. By the same argument $\beta_3 = 1$, since there has to be at least one individual who suffers from the project. The numerator is maximal if $\alpha_2 = \frac{N+1}{2}$ and $\alpha_1 = 0$. Due to the equilibrium refinement β_1 has to be zero since otherwise individuals with the highest benefit would have voted against the proposal. It follows that $\beta_2 = \frac{N-1}{2} - 1$. We obtain

$$\begin{array}{rrcl} & \max(\frac{A_W}{A_L}) & = & \frac{N-1}{N+1} \frac{\frac{N+1}{2} e^{-r\gamma\epsilon}}{\frac{N-3}{2} e^{-r\gamma\epsilon} + e^{r\gamma\epsilon}} \\ & & = & \frac{N-1}{N-3+2e^{2r\gamma\epsilon}} \\ & & \leq & \sqrt{2} - 1 \\ \Leftrightarrow & N-1 & \leq & (\sqrt{2}-1)(N-3) + 2(\sqrt{2}-1)e^{2r\gamma\epsilon} \\ \Leftrightarrow & \frac{(2-\sqrt{2})N-4+3\sqrt{2}}{2(\sqrt{2}-1)} & \leq & e^{2r\gamma\epsilon} \\ \Leftrightarrow & \frac{1}{2\epsilon\gamma} \log\Big(\frac{N}{\sqrt{2}} + \frac{3\sqrt{2}-4}{2(\sqrt{2}-1)}\Big) & \leq & r \end{array}$$

Since we are looking for the smallest $\hat{R}$ such that MV is better than SM if risk aversion r is greater than $\hat{R}$ we obtain $\hat{R} = \frac{1}{2\epsilon\gamma} \log\Big(\frac{N}{\sqrt{2}} + \frac{3\sqrt{2}-4}{2(\sqrt{2}-1)}\Big)$.

■

Proof of Proposition 5

We use the already introduced notation $A(z_{i1}) = F(z_{i1}, 1) - 2F(z_{i1}, 0) + F(z_{i1}, -1)$. The difference between the aggregated expected utilities is given by

$$\begin{aligned} & \Delta\mathbb{E}_1[W_2^{MV}] - \Delta\mathbb{E}_1[W_2^{SM}] \\ = \ & \frac{1}{4^N}\Big\{ \sum_{\beta_1=0}^{\frac{N-1}{2}} \sum_{\beta_2=0}^{\frac{N-1}{2}-\beta_1} \sum_{\alpha=0}^{\frac{N-1}{2}-\beta_1-\beta_2} \binom{N}{\beta_1}\binom{N-\beta_1}{\beta_2}\binom{N-\beta_1-\beta_2}{\alpha} \\ & \cdot\Big[(P(\frac{N-1}{2}) - P(N))[\beta_1 A(-\gamma) + \beta_2 A(-1) + \alpha A(\gamma) + (\frac{N-1}{2} - \alpha - \beta_1 - \beta_2)A(1)] \\ & \qquad + (\frac{1}{2} - P(N))\frac{N+1}{2}A(1)\Big] \\ & + \sum_{\beta_1=0}^{\frac{N-1}{2}} \sum_{\beta_2=0}^{\frac{N-1}{2}-\beta_1} \sum_{\alpha=\frac{N+1}{2}-\beta_1-\beta_2}^{N-\beta_1-\beta_2} \binom{N}{\beta_1}\binom{N-\beta_1}{\beta_2}\binom{N-\beta_1-\beta_2}{\alpha} \\ & \cdot\Big[(P(\frac{N-1}{2}) - P(N))[\beta_1 A(-\gamma) + \beta_2 A(-1) + (\frac{N-1}{2} - \beta_1 - \beta_2)A(\gamma)] \\ & \qquad + (\frac{1}{2} - P(N))[(\alpha - \frac{N-1}{2} + \beta_1 + \beta_2)A(\gamma) + (N - \alpha - \beta_1 - \beta_2)A(1)]\Big]\Big\} \\ & + \frac{1}{2}A(0)[\frac{N-1}{2}P(\frac{N-1}{2}) - NP(N) + \frac{N+1}{4}] \end{aligned} \tag{6}$$

Note that $\frac{1}{2} - P(N) < 0$, $P(\frac{N-1}{2}) - P(N) > 0$ and $\frac{N-1}{2}P(\frac{N-1}{2}) - NP(N) + \frac{N+1}{4} < 0$ (see proof of Proposition 3). Rearranging terms yields

$$
\begin{aligned}
& \Delta\mathbb{E}_1[W_2^{MV}] \geq \Delta\mathbb{E}_1[W_2^{SM}] \\
\Leftrightarrow \quad & \Big[P(\tfrac{N-1}{2}) - P(N)\Big]\tfrac{1}{4^N}\Big(\textstyle\sum_{\beta_1=0}^{\frac{N-1}{2}}\sum_{\beta_2=0}^{\frac{N-1}{2}-\beta_1}\sum_{\alpha=0}^{\frac{N-1}{2}-\beta_1-\beta_2}\binom{N}{\beta_1}\binom{N-\beta_1}{\beta_2}\binom{N-\beta_1-\beta_2}{\alpha} \\
& \qquad \cdot[\beta_1 A(-\gamma) + \beta_2 A(-1) + \alpha A(\gamma) + (\tfrac{N-1}{2} - \alpha - \beta_1 - \beta_2)A(1)] \\
& \quad + \textstyle\sum_{\beta_1=0}^{\frac{N-1}{2}}\sum_{\beta_2=0}^{\frac{N-1}{2}-\beta_1}\sum_{\alpha=\frac{N+1}{2}-\beta_1-\beta_2}^{N-\beta_1-\beta_2}\binom{N}{\beta_1}\binom{N-\beta_1}{\beta_2}\binom{N-\beta_1-\beta_2}{\alpha} \\
& \qquad \cdot[\beta_1 A(-\gamma) + \beta_2 A(-1) + (\tfrac{N-1}{2} - \beta_1 - \beta_2)A(\gamma)]\Big) \\
& \qquad\qquad \geq \\
& \Big[P(N) - \tfrac{1}{2}\Big]\tfrac{1}{4^N}\Big(\textstyle\sum_{\beta_1=0}^{\frac{N-1}{2}}\sum_{\beta_2=0}^{\frac{N-1}{2}-\beta_1}\sum_{\alpha=0}^{\frac{N-1}{2}-\beta_1-\beta_2}\binom{N}{\beta_1}\binom{N-\beta_1}{\beta_2}\binom{N-\beta_1-\beta_2}{\alpha}\tfrac{N+1}{2}A(1) \\
& \quad + \textstyle\sum_{\beta_1=0}^{\frac{N-1}{2}}\sum_{\beta_2=0}^{\frac{N-1}{2}-\beta_1}\sum_{\alpha=\frac{N+1}{2}-\beta_1-\beta_2}^{N-\beta_1-\beta_2}\binom{N}{\beta_1}\binom{N-\beta_1}{\beta_2}\binom{N-\beta_1-\beta_2}{\alpha} \\
& \qquad \cdot[(\alpha - \tfrac{N-1}{2} + \beta_1 + \beta_2)A(\gamma) + (N - \alpha - \beta_1 - \beta_2)A(1)]\Big) \\
& + [NP(N) - \tfrac{N-1}{2}P(\tfrac{N-1}{2}) - \tfrac{N+1}{4}]\tfrac{1}{2}A(0)
\end{aligned}
$$

All coefficients are positive (on both sides of this inequality). We also know that $A(z_{i1})$ is positive for all $z_{i1} \in \{-1, -\gamma, \gamma, 1\}$. To show that there is an R such that this inequality holds for all $r \geq R$, we first multiply the inequality by e^{-r} and then evaluate the convergence properties of each term, i.e. each $A(z_{i1})$ separately. Note that on the RHS there are only terms with $A(0)$, $A(\gamma)$ and $A(1)$.

$$
\begin{array}{rcll}
e^{-r}A(-1) & = & \frac{1}{r}(e^{r(\epsilon-2)} - 2e^{r(\epsilon-1)} + e^{r\epsilon}) & \to \infty \quad (r \to \infty) \\
e^{-r}A(-\gamma) & = & \frac{1}{r}(e^{r(\epsilon\gamma-2)} - 2e^{r(\epsilon\gamma-1)} + e^{r\epsilon\gamma}) & \to \infty \quad (r \to \infty) \\
e^{-r}A(0) & = & \frac{1}{r}(e^{-2r} - 2e^{-r} + 1) & \to 0 \quad (r \to \infty) \\
e^{-r}A(\gamma) & = & \frac{1}{r}(e^{r(-\epsilon\gamma-2)} - 2e^{r(-\epsilon\gamma-1)} + e^{-r\epsilon\gamma}) & \to 0 \quad (r \to \infty) \\
e^{-r}A(1) & = & \frac{1}{r}(e^{r(-\epsilon-2)} - 2e^{r(-\epsilon-1)} + e^{-r\epsilon}) & \to 0 \quad (r \to \infty)
\end{array}
$$

The RHS multiplied by e^{-r} tends to zero as r goes to infinity, while the LHS multiplied by e^{-r} tends to infinity. That means that there exists a smallest $R \in \mathbb{R}$ such that the LHS is greater than the RHS.

■

Proof of Lemma 4

The constants are given by

$$
\begin{aligned}
C &= \tfrac{1}{4^N} \\
c_1 &= \textstyle\sum_{\beta_1=0}^{\frac{N-1}{2}} \sum_{\beta_2=0}^{\frac{N-1}{2}-\beta_1} \sum_{\alpha=0}^{\frac{N-1}{2}-\beta_1-\beta_2} \binom{N}{\beta_1}\binom{N-\beta_1}{\beta_2}\binom{N-\beta_1-\beta_2}{\alpha} \\
&\quad \cdot\Big[(P(\tfrac{N-1}{2})-P(N))(\tfrac{N-1}{2}-\alpha-\beta_1-\beta_2)+(\tfrac{1}{2}-P(N))\tfrac{N+1}{2}\Big] \\
&\quad +\textstyle\sum_{\beta_1=0}^{\frac{N-1}{2}} \sum_{\beta_2=0}^{\frac{N-1}{2}-\beta_1} \sum_{\alpha=\frac{N+1}{2}-\beta_1-\beta_2}^{N-\beta_1-\beta_2} \binom{N}{\beta_1}\binom{N-\beta_1}{\beta_2}\binom{N-\beta_1-\beta_2}{\alpha}(\tfrac{1}{2}-P(N))(N-\alpha-\beta_1-\beta_2) \\
c_2 &= \textstyle\sum_{\beta_1=0}^{\frac{N-1}{2}} \sum_{\beta_2=0}^{\frac{N-1}{2}-\beta_1} \sum_{\alpha=0}^{\frac{N-1}{2}-\beta_1-\beta_2} \binom{N}{\beta_1}\binom{N-\beta_1}{\beta_2}\binom{N-\beta_1-\beta_2}{\alpha}(P(\tfrac{N-1}{2})-P(N))\alpha \\
&\quad +\textstyle\sum_{\beta_1=0}^{\frac{N-1}{2}} \sum_{\beta_2=0}^{\frac{N-1}{2}-\beta_1} \sum_{\alpha=\frac{N+1}{2}-\beta_1-\beta_2}^{N-\beta_1-\beta_2} \binom{N}{\beta_1}\binom{N-\beta_1}{\beta_2}\binom{N-\beta_1-\beta_2}{\alpha} \\
&\quad \cdot\Big[(P(\tfrac{N-1}{2})-P(N))(\tfrac{N-1}{2}-\beta_1-\beta_2)+(\tfrac{1}{2}-P(N))(\alpha-\tfrac{N-1}{2}+\beta_1+\beta_2)\Big] \\
c_3 &= \textstyle\sum_{\beta_1=0}^{\frac{N-1}{2}} \sum_{\beta_2=0}^{\frac{N-1}{2}-\beta_1} \sum_{\alpha=0}^{\frac{N-1}{2}-\beta_1-\beta_2} \binom{N}{\beta_1}\binom{N-\beta_1}{\beta_2}\binom{N-\beta_1-\beta_2}{\alpha}(P(\tfrac{N-1}{2})-P(N))\beta_2 \\
&\quad +\textstyle\sum_{\beta_1=0}^{\frac{N-1}{2}} \sum_{\beta_2=0}^{\frac{N-1}{2}-\beta_1} \sum_{\alpha=\frac{N+1}{2}-\beta_1-\beta_2}^{N-\beta_1-\beta_2} \binom{N}{\beta_1}\binom{N-\beta_1}{\beta_2}\binom{N-\beta_1-\beta_2}{\alpha}(P(\tfrac{N-1}{2})-P(N))\beta_2 \\
c_4 &= \textstyle\sum_{\beta_1=0}^{\frac{N-1}{2}} \sum_{\beta_2=0}^{\frac{N-1}{2}-\beta_1} \sum_{\alpha=0}^{\frac{N-1}{2}-\beta_1-\beta_2} \binom{N}{\beta_1}\binom{N-\beta_1}{\beta_2}\binom{N-\beta_1-\beta_2}{\alpha}(P(\tfrac{N-1}{2})-P(N))\beta_1 \\
&\quad +\textstyle\sum_{\beta_1=0}^{\frac{N-1}{2}} \sum_{\beta_2=0}^{\frac{N-1}{2}-\beta_1} \sum_{\alpha=\frac{N+1}{2}-\beta_1-\beta_2}^{N-\beta_1-\beta_2} \binom{N}{\beta_1}\binom{N-\beta_1}{\beta_2}\binom{N-\beta_1-\beta_2}{\alpha}(P(\tfrac{N-1}{2})-P(N))\beta_1 \\
c_5 &= \tfrac{4^N}{2}[\tfrac{N-1}{2}P(\tfrac{N-1}{2})-NP(N)+\tfrac{N+1}{4}]
\end{aligned}
$$

We immediately obtain C, c_3 and c_4 as positive since $P(\frac{N-1}{2})-P(N)>0$. Secondly, $c_5<0$ because $\frac{N-1}{2}P(\frac{N-1}{2})-NP(N)+\frac{N+1}{4}<0$. We now analyze the first summand that determines c_1. Note that the second summand is negative as $\frac{1}{2}-P(N)<0$. The first sum amounts to

$$\tfrac{N-1}{2}P(\tfrac{N-1}{2})-NP(N)+\tfrac{N+1}{4}-(\alpha+\beta_1+\beta_2)P(\tfrac{N-1}{2})<0.$$

Therefore c_1 is also proved to be negative. The relations between the coefficients are obvious: $c_3 = c_4$ since the expected welfare gain of project losers when they form a minority is the same (one can switch the role of β_1 and β_2 without changing the results). Individuals with $z_{i1}=\gamma$ expect a higher utility gain through strategic voting than individuals with $z_{i1}=1$ because of our MM assumption, but they also realize utility losses by belonging to the voting winners. We obtain $c_1<c_2<c_3$.

■

Proof of Lemma 5

The factor $\frac{1}{r}(e^{-r} + e^{r} - 2)$ is positive and monotonically increasing for all $r > 0$. We set

$$H(r) := c_1 e^{-r\epsilon} + c_2 e^{-r\epsilon\gamma} + c_3 e^{r\epsilon} + c_4 e^{r\epsilon\gamma} + c_5.$$

Derivation with respect to r yields

$$\frac{\partial H(r)}{\partial r} = -c_1 \epsilon e^{-r\epsilon} - c_2 \epsilon\gamma e^{-r\epsilon\gamma} + c_3 \epsilon e^{r\epsilon} + c_4 \epsilon\gamma e^{r\epsilon\gamma}.$$

Note that

$$\begin{aligned} -c_2 \gamma e^{-r\epsilon\gamma} + c_4 \gamma e^{r\epsilon\gamma} &> 0 \\ \Leftrightarrow \qquad c_4 e^{2r\epsilon\gamma} &> c_2 \end{aligned}$$

which is fulfilled since $e^x > 1 \; \forall \, x > 0$ and $c_4 > c_2$. Using Lemma 4 and $\epsilon, \gamma > 0$ yields $\frac{\partial H(r)}{\partial r} > 0$. Therefore $\Delta\mathbb{E}_1[W_2^{MV}] - \Delta\mathbb{E}_1[W_2^{SM}]$ is monotonically increasing in r as soon as $H(r) \geq 0$.

■

9 Appendix B

Sample Values of $\hat{R}(N, \epsilon, \gamma)$

- The first tables contain $\hat{R}(N, \epsilon, \gamma)$ for large N with $N \equiv 3(4)$.

  Table 1a: $\epsilon = 1/2$.

γ/N	11	23	51	103	1003
$\frac{1}{4}$	8.35	11.23	14.37	17.17	26.26
$\frac{1}{2}$	4.18	5.61	7.19	8.58	13.13
$\frac{3}{4}$	2.78	3.74	4.79	5.72	8.75

  Table 1b: $\gamma = 1/4$.

ϵ/N	11	23	51	103	1003
$\frac{1}{10}$	41.77	56.14	71.87	85.84	131.29
$\frac{1}{2}$	8.35	11.23	14.37	17.17	26.26
$\frac{3}{4}$	5.57	7.49	9.58	11.45	17.51
1	4.18	5.61	7.19	8.58	13.13

- The plot in section 4.3 shows that $Q(N) >> \sqrt{2} - 1$ if N is small and $N \equiv 3(4)$. For these numbers of committee members we want to calculate the 'real' R, e.g. the minimal parameter $R(N)$ such that $(\frac{A_W}{A_L})_{max} = Q(N)$. This gives us the following values:

  Table 2a: $\epsilon = \frac{1}{2}$.

γ/N	3	7	11	15	19	23	27	31
$\frac{1}{4}$	2.77	6.16	8.13	9.32	10.3	11.08	11.72	12.28
$\frac{1}{2}$	1.39	3.08	4.06	4.66	5.15	5.54	5.86	6.14
$\frac{3}{4}$	0.92	2.05	2.71	3.11	3.43	3.69	3.91	4.09

  Table 2b: $\gamma = \frac{1}{4}$.

ϵ/N	3	7	11	15	19	23	27	31
$\frac{1}{10}$	13.86	30.81	40.64	46.6	51.49	55.38	58.61	61.4
$\frac{1}{2}$	2.77	6.16	8.13	9.32	10.3	11.08	11.72	12.28
$\frac{3}{4}$	1.85	4.11	5.42	6.21	6.87	7.38	7.81	8.19
1	1.39	3.08	4.06	4.66	5.15	5.54	5.86	6.14

  A comparison with the first tables 1(a,b) for $N = 11$ and $N = 23$ shows that the values do not deviate strongly from $\hat{R}(N, \epsilon, \gamma)$, which indicates that $\hat{R}(N, \epsilon, \gamma)$

is a good approximation even for small N. The 'real' R is smaller than $\hat{R}$, as expected.

- The final tables contain the value of the parameter r such that $\max \frac{A_W}{A_L} = Q(N)$ for small N with $N \equiv 1(4)$.

Table 3a: $\epsilon = \frac{1}{2}$.

γ / N	5	9	13	17	21	25	29
$\frac{1}{4}$	8.79	9.52	10.32	11.05	11.73	12.26	12.85
$\frac{1}{2}$	4.39	4.76	5.16	5.53	5.87	6.13	6.42
$\frac{3}{4}$	2.93	3.17	3.44	3.68	3.91	4.09	4.28

Table 3b: $\gamma = \frac{1}{4}$.

ϵ / N	5	9	13	17	21	25	29
$\frac{1}{10}$	43.94	47.58	51.58	55.27	58.65	61.3	64.25
$\frac{1}{2}$	8.79	9.52	10.32	11.05	11.73	12.26	12.85
$\frac{3}{4}$	5.86	6.34	6.88	7.37	7.82	8.17	8.57
1	4.39	4.76	5.16	5.53	5.87	6.13	6.42

10 Appendix C

N = 5

Suppose $N = 5$. The equality that determines R^* is given by

$$-2572e^{-r\epsilon} - 1812e^{-r\epsilon\gamma} + 400e^{r\epsilon} + 400e^{r\epsilon\gamma} - 3584 = 0$$

[8]Again we give a sample plot for R^* with $\epsilon \in \{0.1, 0.5, 0.9\}$.

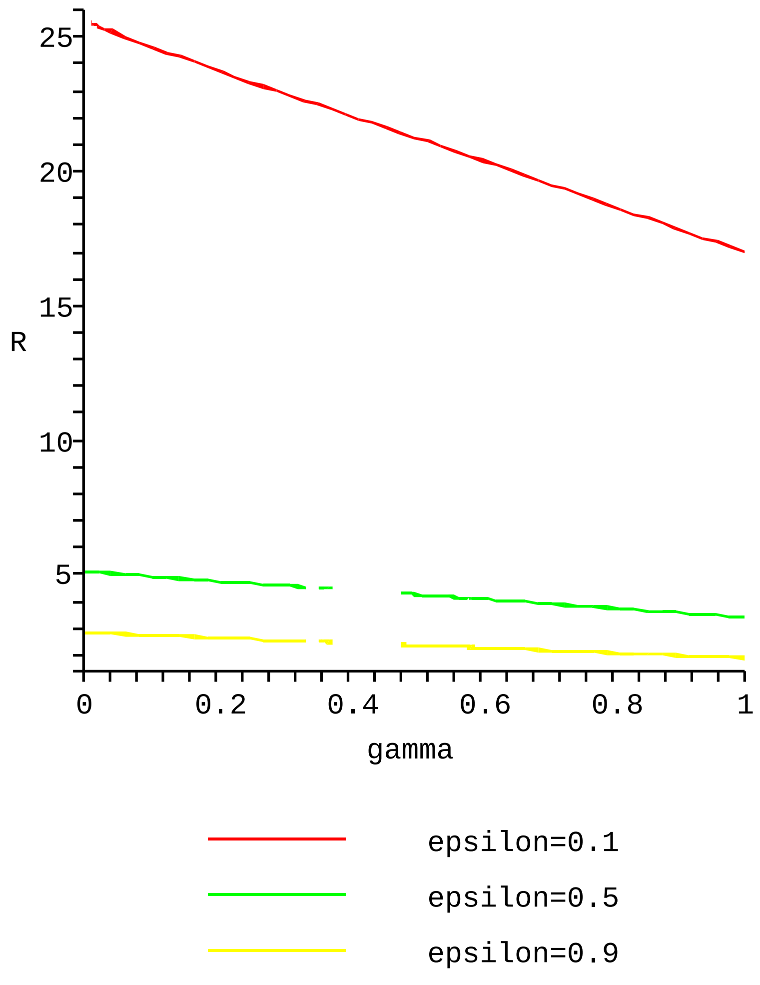

Figure 3: $N = 5$, $\epsilon = 0.1$, $\gamma \in [0, 1]$

Again, the upper line indicates the case where $\epsilon = 0.1$, the middle line represents $\epsilon = 0.5$, and the lower line uses $\epsilon = 0.9$.

[8]If $N = 5$ then $\Delta\mathbb{E}_1[W_2^{MV}] - \Delta\mathbb{E}_1[W_2^{SM}] = \frac{1}{4^5\, r} \cdot \frac{1}{16}(e^{-r} + e^{r} - 2)[-2572e^{-r\epsilon} - 1812e^{-r\epsilon\gamma} + 400e^{r\epsilon} + 400e^{r\epsilon\gamma} - 3584]$.

N = 7

Suppose $N = 7$. The equality that determines R^* is given by

$$-92208e^{-r\epsilon} - 57040e^{-r\epsilon\gamma} + 29568e^{r\epsilon} + 29568e^{r\epsilon\gamma} - 90112 = 0$$

[9]Again we give samples for R^*.

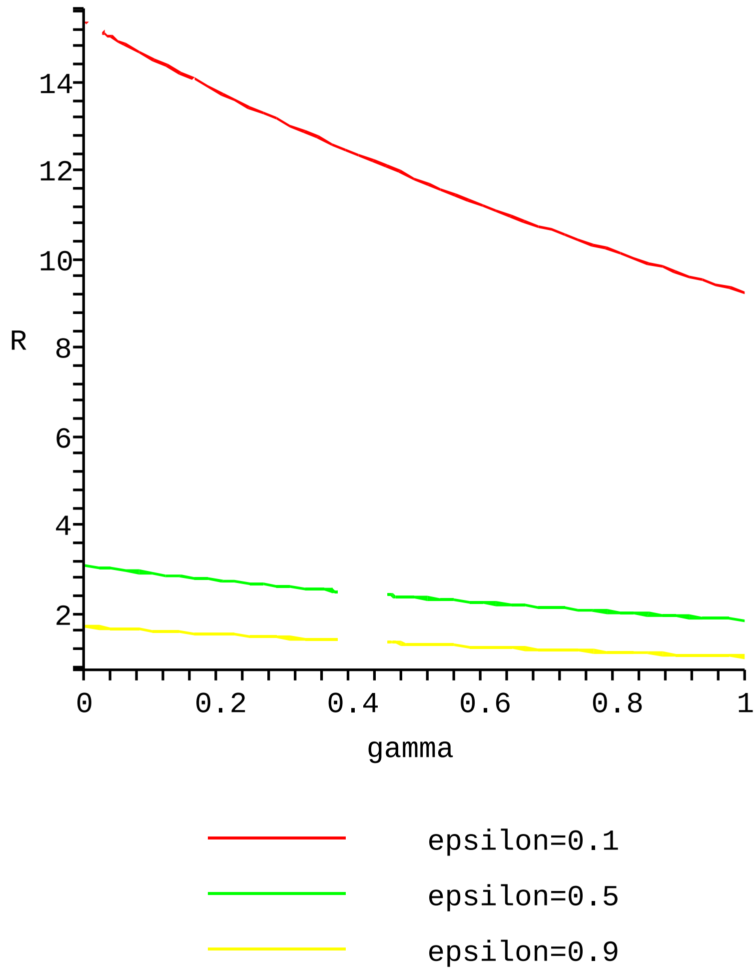

Figure 4: $N = 7$, $\epsilon = 0.1$, $\gamma \in [0, 1]$

[9]If $N = 7$ then $\Delta\mathbb{E}_1[W_2^{MV}] - \Delta\mathbb{E}_1[W_2^{SM}] = \frac{1}{4^7}\frac{1}{r}\frac{1}{32}(e^{-r} + e^{r} - 2)[-92208e^{-r\epsilon} - 57040e^{-r\epsilon\gamma} + 29568e^{r\epsilon} + 29568e^{r\epsilon\gamma} - 90112]$.

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